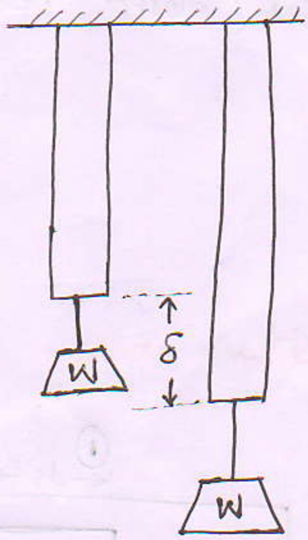
ENERGY THEOREMS

Str. Energy : When a member is deformed under the action of an external loading, the member is said to have stored energy which is called the strain energy of the member or the resilience of the member. S·E stored by a member is equal to amount of work done by the external forces to produce the deformation.

Strain energy ~~dn.~~ stored due to axial loading:



A member of length 'l' & c/s area A is subjected to external axial load W.

Let the extension of member = δ

∵ load is applied gradually; W·D by load on the member equals to product of avg load & the displacement δ.

∴ Ext W·D $W_e = \frac{1}{2} W\delta$

Let energy stored by the member = W_i

W·D by external force = energy stored by it.

$$W_e = W_i$$

Let tension in the member = S

For eq^{bm} of the member ; $S = W$.

Tensile stress $f = \frac{S}{A}$

Tensile strain $e = \frac{f}{E} = \frac{S}{AE}$

Change in length of member δ = strain × original length

$$= e \cdot l$$

$$= \frac{S}{AE} \cdot l$$

S.E stored = Work done

$$= \frac{1}{2} \cdot W \cdot \delta = \frac{1}{2} \cdot S \cdot \frac{S}{AE} l = \frac{S^2 l}{2AE}$$

S.E stored due to axial loading $= \dfrac{S^2 l}{2AE}$

S.E stored per unit volume $= \dfrac{\frac{S^2 l}{2AE}}{Al}$

$$= \frac{S^2 \cancel{l}}{2A^2 E \cancel{l}} = \frac{f^2}{2E}$$

Strain energy stored due to bending.

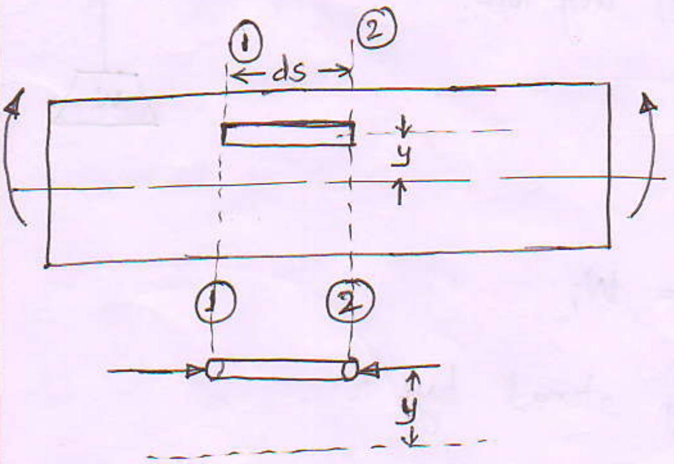


Fig shows a beam subjected to a uniform moment M.

Consider the elemental length 'ds' of the beam betn two sections 1-1 and 2-2; ds

The elemental length of the beam may be considered as consisting of an infinite no. of elemental cylinders each of area 'da' and length 'ds'. Consider one such elemental cylinder situated 'y' units from the neutral layer betⁿ the sections 1-1 & 2-2.

Intensity of stress in the elemental cylinder $f = \frac{M}{I} y$

I : MI of entire section of beam about NA.

Energy stored by elemental cylinder } = Energy stored per unit volume x volume of cylinder

$$= \frac{f^2}{2E} \times da \cdot ds$$

$$= \frac{1}{2E} \left(\frac{M}{I} y \right)^2 \cdot da \cdot ds$$

$$= \frac{M^2}{2EI^2} \cdot da \cdot ds \cdot y^2$$

Energy stored by ds length of beam } = sum of energy stored by each elemental cylinder betⁿ the two sections 1-1 & 2-2

$$= \sum \frac{M^2}{2EI^2} \cdot ds \cdot da \cdot y^2$$

$$= \frac{M^2 \cdot ds}{2EI^2} \sum da \cdot y^2$$

$\sum da \cdot y^2$ = M.I of beam section about the NA.

$= I$.

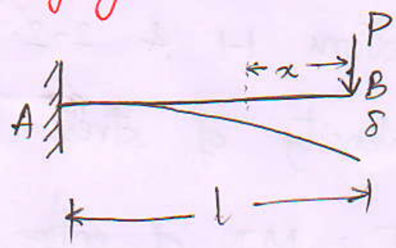
$\therefore$ Energy stored by ds length of beam $= \frac{M^2 \cdot ds}{2EI^2} \cdot I$

$$= \frac{M^2 ds}{2EI}$$

Total energy stored by whole beam $= \int \frac{M^2 \cdot ds}{2EI}$

— x — x —

Prob: Find the defl^n of free end of cantilever carrying @ pt. load P @ its free end.



Sol^n Fig . . . .

δ = defl^n @ B

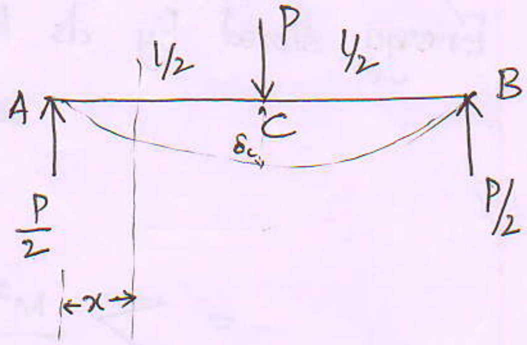
$M = -Px$

S.E stored by cantilever $W_i = \int \frac{M^2 dx}{2EI} = \int_0^l \frac{P^2 x^2 dx}{2EI} = \frac{P^2 l^3}{6EI}$

W.D by external load $= \frac{1}{2} P.\delta$

$$\frac{1}{2} P\delta = \frac{P^2 l^3}{6EI} \Rightarrow \boxed{\delta = \frac{Pl^3}{3EI}}$$

Prob: A ^{ss} beam of span 'l' carries a conc. load P @ its mid span. Find central δ.



Each Reac $= P/2$

BM in AC @ x ~~M~~ $M = \frac{P}{2} x$

S.E stored by the whole beam } $W_i = 2 \times$ strain energy stored by half the beam.

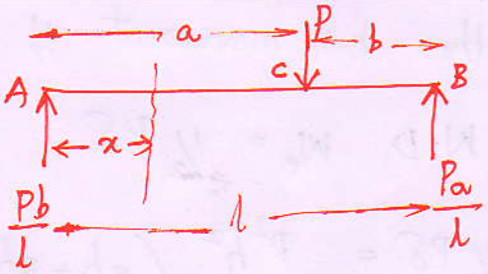
$$= 2\int_0^{l/2} \left(\frac{P}{2}x\right)^2 \cdot \frac{dx}{2EI}$$

$$= \frac{P^2}{4EI}\int_0^{l/2} x^2 dx = \frac{P^2}{4EI}\cdot\frac{1}{3}\cdot\frac{l^3}{8} = \frac{P^2 l^3}{96EI}$$

W.D by external load $= \frac{1}{2} P\delta$

$$\frac{1}{2} P\delta = \frac{P^2 l^3}{96EI} \Rightarrow \boxed{\delta = \frac{Pl^3}{48EI}}$$

Prob: Find defl^n under load



Sol^n: $R_A = \frac{Pb}{l}$; $R_B = \frac{Pa}{l}$

SE stored by beam AB W_i

$= S \cdot E$ stored by AC + $S \cdot E$ stored by BC.

$$= \int_0^a \left(\frac{Pb}{l}x\right)^2 \cdot \frac{dx}{2EI} + \int_0^b \left(\frac{Pa}{l}x\right)^2 \cdot \frac{dx}{2EI}$$

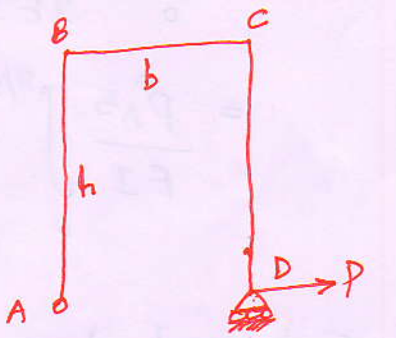
$$= \frac{P^2 b^2 a^3}{6EI l^2} + \frac{Pa^2 b^2}{6EI l^2} = \frac{P^2 a^2 b^2}{6EI l^2}(a+b) = \frac{P^2 a^2 b^2}{6EIl}$$

Defl^n @ $C = \delta$

$W \cdot D = \frac{1}{2} P\delta$

Equating WD to S.E stored; $\frac{1}{2}P\delta = \frac{P^2 a^2 b^2}{6EIl} \Rightarrow$ $\boxed{\delta = \frac{Pa^2 b^2}{3EIl}}$

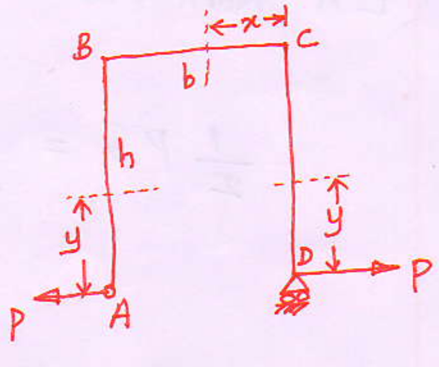
Prob: A portal frame ABCD has its end A hinged while the end D is placed on rollers. A horizontal force 'P' is applied on the end D as shown in fig. Determine the hori movement of D.. Assume all the members to have same EI.



Sol^n: $\Sigma H = 0$

$H_a + P = 0 \Rightarrow H_a = -P \ (\leftarrow)$

S.E stored by frame W_i = S.E stored by columns + SE stored by beams.



$$W_i = 2\int_0^h \frac{P^2 y^2}{2EI} dy + \int_0^b \frac{P^2 h^2 \cdot dx}{2EI}$$

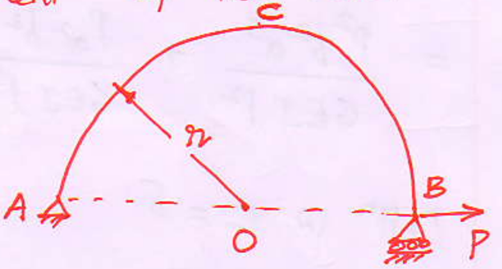
$$= 2 \cdot \frac{P^2 h^3}{6EI} + \frac{P^2 h^2 b}{2EI} = \frac{P^2 h^2}{6EI}(2h + 3b)$$

Let the hori movement of $D = \delta$

Ext W·D $W_e = \frac{1}{2} P\delta$

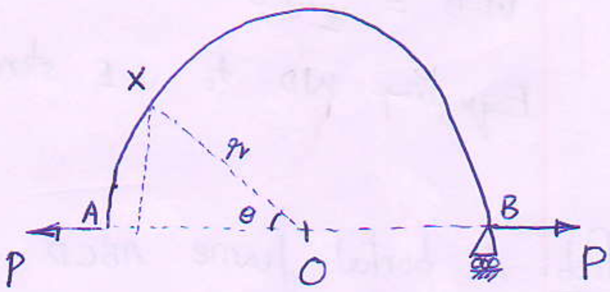
$$\frac{1}{2}P\delta = \frac{P^2 h^2}{6EI}(2h+3b) \Rightarrow \delta = \frac{Ph^2(2h+3b)}{3EI}$$

Prob: One end of the semicircular arch shown in fig is hinged while its other end is placed on rollers. If the roller end is pulled with a hori force 'P' as shown; determine the hori displacement of the roller end. Assume uniform EI.



Soln: Hori Reaction @ A = $P\ (\leftarrow)$

Consider any section 'X',



BM @ section; $M = Pr\sin\theta$

S.E stored by the arch;

$$W_i = \int \frac{M^2 \cdot ds}{2EI}$$

$ds = r\,d\theta$

$$= 2\int_0^{\pi/2} \frac{P^2 r^2 \sin^2\theta}{2EI}\, r\cdot d\theta$$

$$= \frac{P^2 r^3}{EI}\int_0^{\pi/2} \sin^2\theta \cdot d\theta = \frac{P^2 r^3}{EI}\left[\int_0^{\pi/2} \frac{(1-\cos 2\theta)}{2}\, d\theta\right] = \frac{Pr^3}{EI}\left[\frac{1}{2}\int_0^{\pi/2} d\theta\right.$$

$$= \frac{P^2 r^3}{EI} \times \frac{\pi}{4}$$

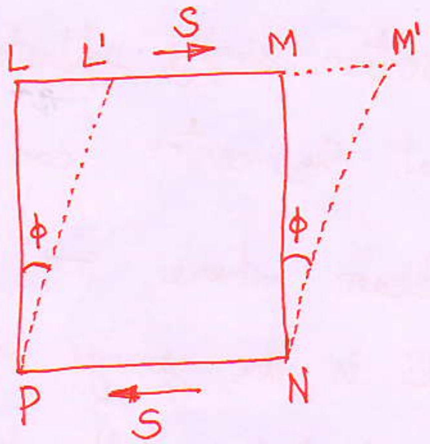
$$\boxed{\begin{aligned} &\frac{1}{2}\int_0^{\pi/2} d\theta - \frac{1}{2}\int_0^{\pi/2} \cos 2\theta \cdot d\theta \\ &\frac{1}{2}\left[\theta\right]_0^{\pi/2} - \frac{1}{2}\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/2} \\ &\frac{1}{2}\left[(\pi/2 - 0) - (0-0)\right] \\ &= \pi/4 \end{aligned}}$$

Ext work done $= \frac{1}{2}P\delta$

$$\frac{1}{2}P\delta = \frac{\pi}{4EI}P^2 r^3 \Rightarrow \delta = \frac{\pi P r^3}{2EI}$$

Strain energy in pure shearing

Consider a rectangular block of material subjected to shearing forces 'S' acting across two of its opposite faces.



The face LM will move, relative to face NP, ~~ay~~

Distance $MM' = MN \times \phi$ $\quad$ ϕ : angle of shear

Work done $= \frac{1}{2} \times S \times MM'$

$= \frac{1}{2} \times S \times MN \times \phi$

$\frac{MM'}{MN} = \tan\phi = \phi$ $\quad$ ($\because$ ϕ is very small)

Now shearing stress $\tau = \frac{S}{LM \times 1}$ $\Rightarrow$ $S = \tau \cdot LM$

Shear strain $\phi = \frac{\tau}{C}$

Work done $= \frac{1}{2} \times S \times MN \times \phi$

$= \frac{1}{2}(\tau \cdot LM) \cdot MN \cdot \frac{\tau}{C}$

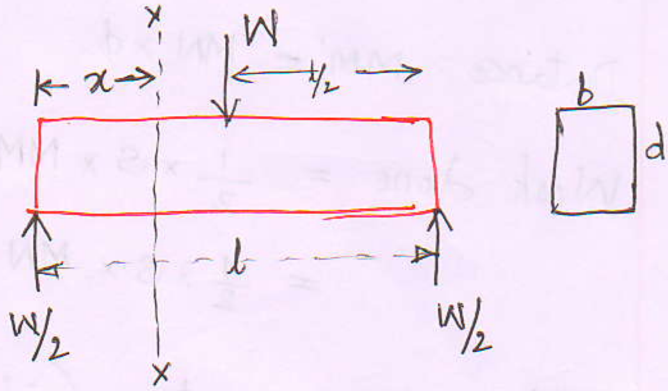
$= \frac{\tau^2}{2C}(LM \times MN \times 1)$

$= \frac{\tau^2}{2C} \times$ volume of block $\quad$ (Taking unit depth normal to diagram)

Strain energy = Work done

$$\boxed{U = \frac{\tau^2}{2C} \times \text{volume of block}}$$

Prob: S.T the S.E U due to ~~beeg~~ bending of a beam of rectangular section simply supported at the ends with a conc load W at the centre can be expressed as $U = \frac{\sigma^2}{18E} \times$ volume of the beam where σ is the bending stress in the beam and E is the youngs Modulus. Compare the S.E when the beam is loaded axially by load W, if the ratio of length to depth of beam is 6.



Sol^n $M = +\frac{W}{2}x$

S.E for left half of beam,

$$= \int \frac{M^2 dx}{2EI}$$

$$= \int_0^{l/2} \frac{\left(\frac{W}{2}x\right)^2 dx}{2EI} = \frac{W}{4EI}\int_0^{l/2} x^2 dx = \frac{W^2 l^3}{192EI}$$

S.E for the whole beam $U = 2 \times \frac{W^2 l^3}{192EI} \Rightarrow \boxed{U = \frac{W^2 l^3}{96EI}}$ — 1

$$\sigma = \frac{M_{max}}{Z} = \frac{\frac{Wl}{4}}{\frac{bd^2}{6}} = \frac{3Wl}{2bd^2} \Rightarrow \boxed{W = \frac{2\sigma bd^2}{3l}}$$

Substituting the value of 'W' in 1

$$U = \left(\frac{2\sigma bd^2}{3l}\right)^2 \frac{l^3}{96EI} = \frac{4\sigma^2 b^2 d^4}{9l^2} \times \frac{l^3}{96 \times E \times \frac{bd^3}{12}}$$

$$= \frac{\sigma^2}{18E}(lbd) = \frac{\sigma^2}{18E} \times \text{vol of beam.}$$

Comparison of strain energies.

U_a = S.E of beam under axial load

$$= \frac{\sigma^2}{2E} \times \text{volume}$$

$$= \frac{1}{2E}\left[\frac{W}{A}\right]^2 \times l \times b \times d$$

$$= \frac{1}{2E} \times \frac{W^2}{b^2 \cdot d^2} \cdot l \cdot b \cdot d = \frac{W^2 l}{2E\,bd} \quad \text{—— } (2)$$

Dividing (2) by (1);

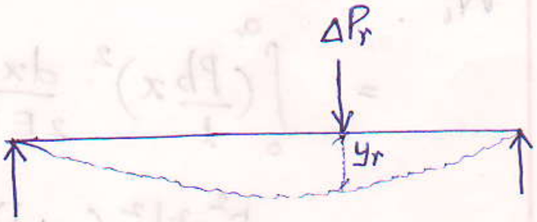
$$\frac{U_a}{U} = \frac{W^2 l}{2E\,bd} \times \frac{96\,EI}{W^2 l^3} = \frac{48\,I}{bd\,l^2} = \frac{48 \cdot \frac{bd^3}{12}}{bd\,l^2}$$

$$\frac{U_a}{U} = 4\left(\frac{d}{l}\right)^2 = 4\left(\frac{1}{6}\right)^2 = 1/9$$

$$\boxed{\frac{U_a}{U} = \frac{1}{9}}$$

Prob: First theorem of Castigliano.

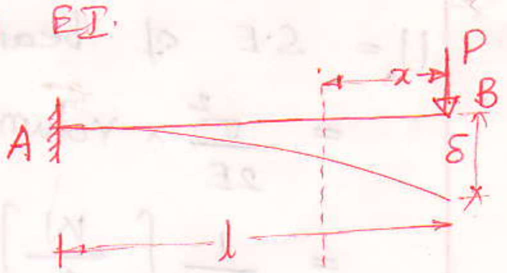
In any beam subjected to any load system; defl^n @ any point 'r' is given by the partial differential co-efficient of the total strain energy stored w.r.t force P_r acting at the point 'r' in the dir^n in which the defl^n is desired.



$$y_r = \frac{\partial W_i}{\partial P_r}$$

= Partial differential co-efficient of the total str. Eng stored w.r.t P_r.

Prob: Find the defl^n at the free end of a cantilever carrying a P.L @ the free end. Assume uniform EI.



Sol^n Fig shows ------

$M = -Px$.

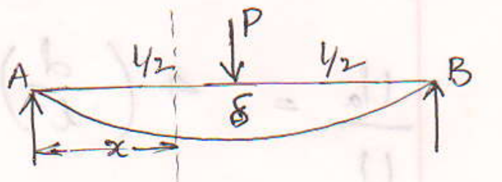
S.E stored by cantilever $W_i = \int \frac{M^2 \cdot dx}{2EI}$

$= \int_0^l \frac{P^2x^2}{2EI} \cdot dx = \frac{P^2}{2EI} \cdot \frac{l^3}{3}$

from 1st theorom of Castigliano; $\delta = \frac{\partial W_i}{\partial P}$

$$\delta = \frac{2P}{2EI} \cdot \frac{l^3}{3} = \frac{Pl^3}{3EI}$$

Prob: Using Castigliano's theorem; find central defl^n of SSB carrying P.L @ mid span

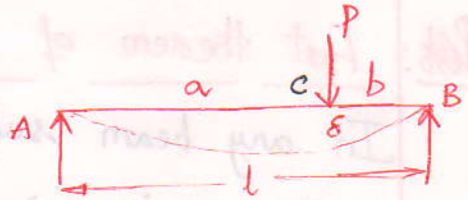


$M_x = P/_2 \, x$

$$W_i = 2\int_0^{l/2} \frac{(P/_2 x)^2 \, dx}{2EI} = \frac{P^2 l^3}{96EI}$$

$$\delta = \frac{\partial W_i}{\partial P} = \frac{2P.l^3}{96EI} = \frac{Pl^3}{48EI}$$

Prob: Find δ using first theorem of Castig..



$R_A = \frac{Pb}{l}$; $R_B = \frac{Pa}{l}$

W_i = S.E stored by AC + S.E stored by BC

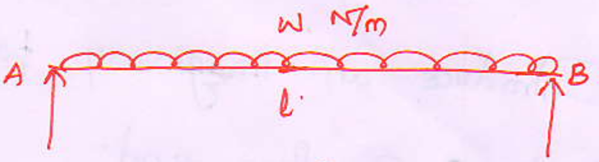
$$= \int_0^a \left(\frac{Pb}{l}x\right)^2 \cdot \frac{dx}{2EI} + \int_0^b \left(\frac{Pa}{l}x\right)^2 \cdot \frac{dx}{2EI}$$

$$= \frac{P^2a^2b^2}{6EIl^2}(a+b) = \frac{P^2a^2b^2}{6EIl}$$

$$\delta = \frac{\partial W_i}{\partial P} = \frac{(2P)a^2b^2}{6EIl} = \frac{Pa^2b^2}{3EIl}.$$

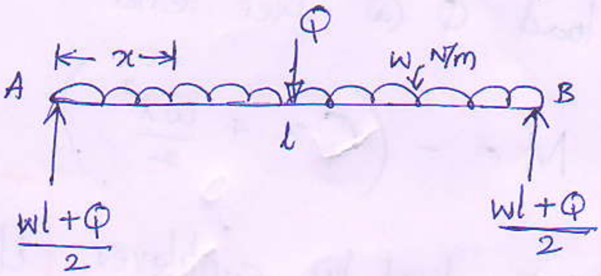
Prob: Find the defl^n @ centre of beam carrying udl as shown.

Sol^n: Introduce an imaginary PL Q @ center.



$$R_A = R_B = \frac{wl + Q}{2}$$

$$BM @ X = \frac{wl+Q}{2}x - \frac{wx^2}{2}$$



Total S.E stored by beam

$$\boxed{U = \int \frac{M^2\, dx}{2EI}}$$

$$U = 2\int_0^{l/2} \left[\frac{wl+Q}{2}x - \frac{wx^2}{2}\right]^2 \frac{dx}{2EI}$$

$$= \frac{1}{EI}\int_0^{l/2} \left(\frac{wl+Q}{2}x - \frac{wx^2}{2}\right)^2 dx$$

To find central defl^n; differentiating total S.E stored w.r.t Q

$$\boxed{\delta = \frac{\partial U}{\partial Q}} = \frac{1}{EI}\int_0^{l/2} 2\left(\frac{wl+Q}{2}x - \frac{wx^2}{2}\right)\frac{x}{2}\cdot dx$$

$$= \frac{1}{EI}\left[\int_0^{l/2} \frac{wl+Q}{2}x^2\, dx - \int_0^{l/2} \frac{wx^3}{2} dx\right]$$

$$= \frac{1}{EI}\left\{\left[\frac{wl+Q}{2}\cdot\frac{x^3}{3}\right]_0^{l/2} - \left[\frac{w}{2}\cdot\frac{x^4}{4}\right]_0^{l/2}\right\}$$

$$= \frac{1}{EI}\left[\frac{wl+Q}{2}\cdot\frac{1}{3}\cdot\frac{l^3}{8} - \frac{w}{2}\cdot\frac{1}{4}\cdot\frac{l^4}{16}\right]$$

$$= \frac{1}{EI}\left[\frac{wl^4}{48} + \frac{Ql^3}{48} - \frac{wl^4}{128}\right] = \frac{1}{EI}\left[\frac{wl^4}{48} - \frac{wl^4}{128} + \frac{Ql^3}{48}\right]$$

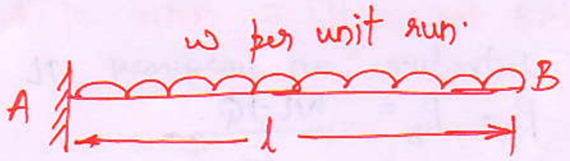
$$= \frac{1}{EI}\left[\frac{5}{384}wl^4 + \frac{Ql^3}{48}\right]$$

Putting $Q = 0$

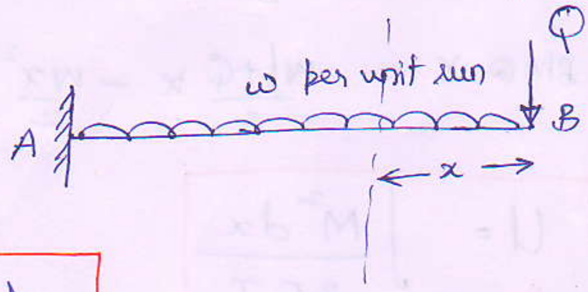
$$\boxed{\delta = \frac{5}{384}\frac{wl^4}{EI}}$$

—— x —— x ——

Prob: Find defl^n @ free end of cantilever



Introduce an imaginary point load Q @ free end.



$$M = -\left(Qx + \frac{wx^2}{2}\right)$$

S.E stored by cantilever $\boxed{U = \int \frac{M^2 \cdot dx}{2EI}}$

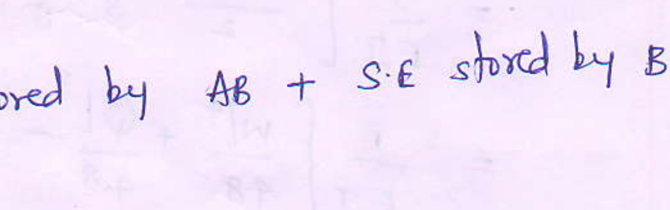
$$U = \int_0^l \left(Qx + \frac{wx^2}{2}\right)^2 \frac{dx}{2EI}$$

$$\boxed{\delta = \frac{\partial U}{\partial Q}} = \int_0^l 2\left(Qx + \frac{wx^2}{2}\right) \cdot \frac{x \cdot dx}{2EI}$$

$$= \frac{1}{EI}\int_0^l \left(Qx^2 + \frac{wx^3}{2}\right) \cdot dx = \frac{1}{EI}\left[\frac{Ql^3}{3} + \frac{wl^4}{8}\right]$$

Putting $Q = 0$; $\boxed{\delta = \frac{wl^4}{8EI}}$

Prob: Find the vertical defl^n of the load W for the beam shown in fig.



T.M. ab A;

$V_b l = W \cdot \frac{3l}{2}$

$V_b = \frac{3W}{2}(\uparrow)$; $V_a = \frac{W}{2}(\downarrow)$

S.E stored by beam = SE stored by AB + S.E stored by BC.

$$U = \sum \int \frac{M^2 \cdot dx}{2EI}$$

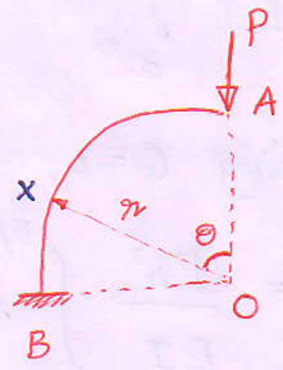
$$= \int_0^{l/2} \left(\frac{W}{2}x\right)^2 \frac{dx}{2EI} + \int_0^{l/2} \frac{(Wx)^2 \cdot dx}{2EI} = \frac{W^2}{8EI} \cdot \frac{l^3}{3} + \frac{W^2}{2EI} \cdot \frac{1}{3} \cdot \frac{l^3}{8}$$

$$U = \frac{W^2 l^3}{16EI}$$

Vertical defl^n $\delta = \frac{\partial U}{\partial W} = \frac{2W}{16}\frac{l^3}{EI} = \frac{Wl^3}{8EI}$

—×—

Prob: The quadrantal ~~arc~~ arch AB shown in fig. Find the vertical & hori defl^{ns} of A.



Sol^n Vertical defl^n of A.

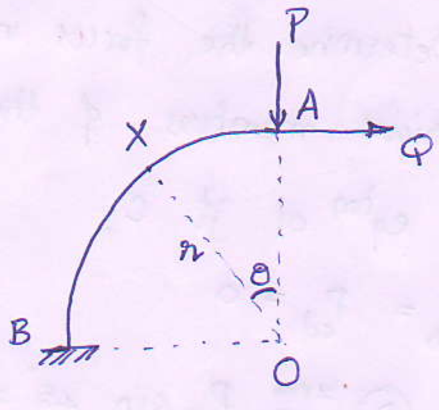
BM @ any section X ; $M = -Pr\sin\theta$

Str. Eng stored $U = \int \frac{M^2 \cdot ds}{2EI}$

$$= \int_0^{\pi/2} \frac{P^2 r^2 \sin^2\theta \cdot r \cdot d\theta}{2EI} = \frac{P^2 r^3}{2EI} \cdot \frac{\pi}{4} = \frac{P^2 \pi r^3}{8EI}$$

Vertical defl^n @ A ; $\delta_v = \frac{\partial U}{\partial P} = \frac{2P \cdot \pi r^3}{8EI} = \frac{P \cdot \pi r^3}{4EI}$

Hori defl^n of A



Introduce an imaginary hori force Q @ A.

BM @ any section X ;

$$M = -\left\{ Pr\sin\theta + Qr(1-\cos\theta) \right\}$$

S.E stored by st^r $U = \int \frac{M^2 \cdot ds}{2EI}$

$$U = \int_0^{\pi/2} \left[Pr\sin\theta + Qr(1-\cos\theta) \right]^2 \cdot \frac{r \cdot d\theta}{2EI}$$

Hori movement of A $\delta_h = \frac{\partial U}{\partial Q}$

$$= \int_0^{\pi/2} 2\left[Pr\sin\theta + Qr(1-\cos\theta)\right] r\,(1-\cos\theta)\cdot \frac{r\cdot d\theta}{2EI}$$

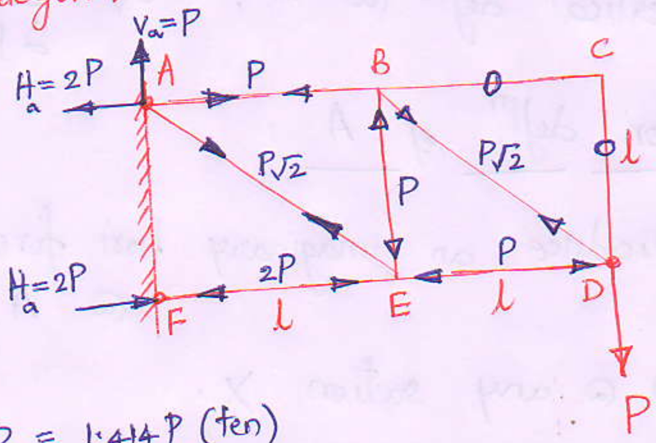
$$= \frac{Pr^3}{EI}\int_0^{\pi/2} \sin\theta\,(1-\cos\theta)\cdot d\theta + \frac{Qr^3}{EI}\int_0^{\pi/2}(1-\cos\theta)^2\cdot d\theta$$

Putting $Q = 0$;

$$\delta_h = \frac{Pr^3}{EI}\int_0^{\pi/2} \sin\theta\,(1-\cos\theta)\cdot d\theta = \frac{Pr^3}{EI}\,\frac{1}{2}\left[(1-\cos\theta)^2\right]_0^{\pi/2}$$

$$\delta_h = \frac{Pr^3}{2EI}$$

Prob: Find the vertical defl^n of joint D of the frame. All the members have same c/s area A. The diagonal members are at 45° with hori.



Sol^n: Determine the forces in the various members of the st^r.

For eq^{bm} of jt C;

$P_{cb} = P_{cd} = 0$

Jt (D), $\Sigma V = 0$; $P_{db}\sin 45^\circ = P \Rightarrow P_{db} = 1.414P$ (ten)

$\Sigma H = 0$: $P_{de} = P\sqrt{2}\times\frac{1}{\sqrt{2}} = P$ (comp^r)

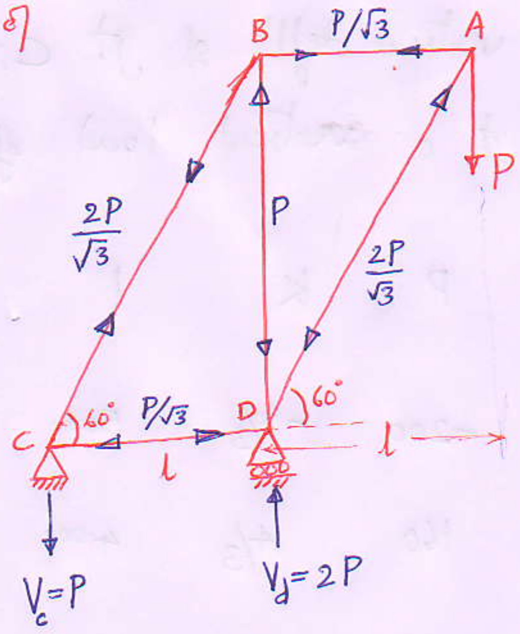
Total S.E stored $W_i = \sum \frac{S^2 l}{2AE}$

= Energy stored by AB, BE, ED + Energy stored by AE, BD + Energy stored by FE

$$= 3.\frac{P^2 l}{2AE} + 2\cdot\frac{(P\sqrt{2})^2\cdot l\sqrt{2}}{2AE} + \frac{4P^2 l}{2AE} = \frac{P^2 l}{2AE}(7+4\sqrt{2})$$

Vertical defl^n @ D $= \frac{\partial W_i}{\partial P} = \frac{2Pl}{2AE}(7+4\sqrt{2}) = \frac{Pl}{AE}(7+4\sqrt{2})$

Prob: Find the vertical defl^n of 'A' of the st^r shown in fig.



Sol^n T.M ab C;

$$P \times 2l = V_d\, l$$

$$V_d = 2P \; ; \quad \therefore V_c = P(\downarrow)$$

1) Jt A
2) Jt B
3) Jt D

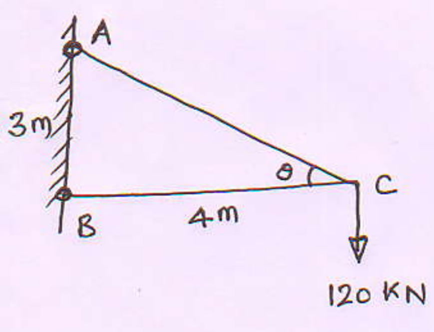
S.E stored by st^r

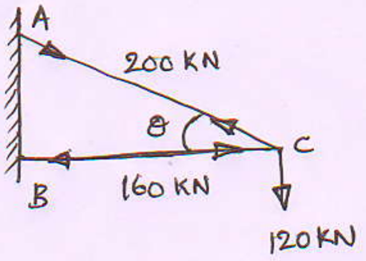
$$W_i = \sum \frac{s^2 l}{2AE} = \text{Eng stored by } AB,\ BC,\ CD;\ DA;\ DB$$

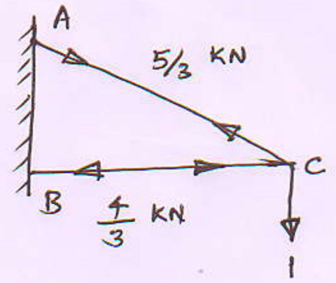
$$= \frac{P^2 l(\sqrt{3}+6)}{2AE}$$

$$\text{Vertical defl}^n \text{ of } A = \frac{\partial W_i}{\partial P} = \frac{Pl}{AE}(\sqrt{3}+6)$$

Prob: Find the vertical ~~& hori~~ defl^n of joint C of truss shown in fig. Area of inclined tie is 2000 mm^2, while area of hori member is 1600 mm^2. $E = 200\ kN/mm^2$







$$\tan\theta = 3/4$$

$$\sin\theta = 3/5$$

$$\cos\theta = 4/5$$

Jt C

$\Sigma V = 0$

$$P_{ca} \sin\theta = 120$$

$$P_{ca} = 200\ KN\ (ten)$$

$\Sigma H = 0$

$$P_{cb} = 160\ KN\ (comp)$$

To find vertical defl^n of jt C; remove given loading & apply a vertical load of 1 kN @ C.

Member	P	K	l	A	$\frac{PKl}{A}$
AC	−200	−5/3	5000	2000	$\frac{2500}{3}$
BC	160	4/3	4000	1600	$\frac{1600}{3}$
				Total	4100/3

Vertical defl^n of C;

$$\sum \frac{PKl}{AE} = \frac{4100}{3 \times 200} = 6.83 \text{ mm}$$