Date
6/1/15

UNIT – I

PROBABILITY

Experiment :– The process of observation is called experiment.

Ex:– Tossing a coin

Types of Experiments :–

1) Predictable experiment

2) Non-predictable experiment (or) Random experiment.

1) **Predictable experiment :–** An experiment is said to be predictable if the result can be predicted. The result is unique

Ex:– Throwing a stone upwards, it would definitely fall to the ground due to gravitational force.

2) **Non-predictable experiment :–** An experiment is said to be random experiment if the results cannot be predicted. The result is not unique.

Ex:– Tossing a coin, getting tail or head.

Sample space :– The collection of all possible outcomes in a random experiment is called sample space. It is denoted by 'S'

Ex:– 1) Tossing a coin, then

$\{H, T\} = 2 = 2^1$

2) Tossing two coins, then

$$S = \{HH, HT, TH, TT\} = 4 = 2^2$$

3) Tossing three coins, then

$$S = \{HHH, HHT, HTH, THT, TTH, HTT, THH, TTT\} = 8 = 2^3$$

NOTE :–

1) In tossing a coin 'n' times, then the sample space consists of 2^n elements.

2) In throwing a dice 'n' times, then the sample space consists of 6^n elements.

Mutually Exclusive events (MEE) :- Events are said to be mutually exclusive if the happening of any one of the events in a trail excludes the happening of any one of the others i.e., if no two or more of the events can happen simultaneously in the same trail. If A and B are mutually exclusive then $A \cap B = \emptyset$ (disjoint sets)

Equally likely events :- Events are said to be equally likely when there is no reason to expect any one of them rather than any one of the others.

Ex:- when a card is drawn from a pack, any card may be obtained.

In this trail all the 52 elementary events are equally likely.

Exhaustive events :- Total no. of all possible outcomes of a random experiment is known as exhaustive events.

Ex:- In tossing a coin there are two exhaustive events.

Favourable events :- The no. of cases favourable to an event in a trail is the no. of outcomes which entail happening of the event.

Ex:- In a throw of 2 dice, the no. of favourable events getting same no. on 2 dice is 6 i.e., {(1,1) (2,2) (3,3) (4,4) (5,5) (6,6)}

Date
1/1/15

Complementary events :- Two events of a sample space whose intersection is $\emptyset$ and whose union is entire sample space are called Complementary events. Thus, if E is any event of sample space 'S' its complement is E' or E^c or $\bar{E}$ and $E \cap E^c = \emptyset$, $E \cup E^c = S$.

Independent events :- Two events are said to be independent if the happening of an event is not effected by the happening of the other event.

Dependent events:- If the occurence of one event effe[cts] the happening of the other event then they are said to be dependent events.

Probability:- In a random experiment, let there be 'n' mutually exclusive and equally likely elementary events. Let 'E' be an event of the experiment. If 'm' favourable events then the probability of E is defined as

$$P(E) = \frac{\text{No. of favourable events}}{\text{Total No. of events}} = \frac{m}{n}$$

In this experiment 'm' events are favourable to E, the remaining n-m events are not favourable to E. It is favourable to $\bar{E}$ or E^c. Then by the definition of probability

$$P(\bar{E}) = \frac{n-m}{n} \qquad P(E) + P(E^c) = 1$$

$$P(E^c) = 1 - \frac{m}{n} = \frac{n-m}{n}$$

$$= 1 - \frac{m}{n}$$

$$= 1 - P(E)$$

$$\therefore P(E) + P(\bar{E}) = 1$$

Axioms of Probability (statements):- In any random experiment, 'S' is a sample space and E is particular event then

i) $0 \leq P(E) \leq 1$

ii) $P(S) = 1$ {sum of probabilities is always one}

iii) If E_1, E_2 are mutually exclusive events (disjoint) then $E_1 \cap E_2 = \phi$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

Theorem-1:- The probability of impossible event is zero i.e., $P(\emptyset) = 0$

Proof:- Let $\emptyset$ is impossible event and S is a sample space.

Now $S \cup \emptyset = S$

Taking probability on both sides

$$P(S \cup \emptyset) = P(S)$$

Here S & $\emptyset$ are Mutually Exclusive events [$\because S \cap \emptyset = \phi$]

$$P(S) + P(\emptyset) = P(S)$$

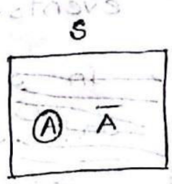
$$P(\emptyset) = 0$$

$\therefore$ The probability of impossible event is zero.

Theorem-2:- If A is any event in 's' then $P(\bar{A}) = 1 - P(A)$

Proof:-

Let A is any event in 's'



i.e., $A \subset S$

Now $A \cup \bar{A} = S$

Taking probability on b.s

$$P(A \cup \bar{A}) = P(S)$$

Here $\bar{A}$ & A are M.E.E

$$P(A) + P(\bar{A}) = P(S)$$

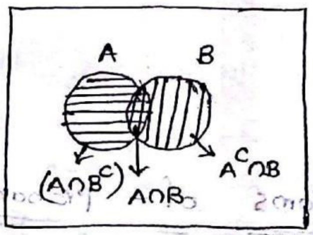
$$P(A) + P(\bar{A}) = 1 \quad (\because \text{from Axiom II})$$

$$\therefore P(\bar{A}) = 1 - P(A)$$

Theorem-3:- If A and B are any two events then $P(A^c \cap B) = P(B) - P(A \cap B)$

Proof:- Let A and B are any two events

Now $B = (A \cap B) \cup (A^c \cap B)$



Taking probability on b.s

$$P(B) = P[(A \cap B) \cup (A^c \cap B)]$$

Here $(A \cap B)$ & $(A^c \cap B)$ are MEE

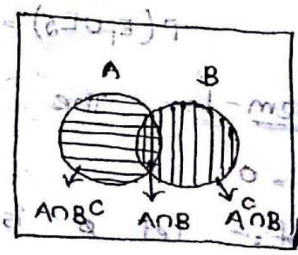
$$P(B) = P(A \cap B) + P(A^c \cap B)$$

$$P(A^c \cap B) = P(B) - P(A \cap B)$$

Theorem-4:- If A and B are any two events then $P(A \cap B^c) = P(A) - P(A \cap B)$.

Proof:- Let A & B are any two events

Now $A = (A \cap B^c) \cup (A \cap B)$



Taking probability on b.s

$$P(A) = P[(A \cap B^c) \cup (A \cap B)]$$

Here $A \cap B^c$ & $A \cap B$ are MEE

$$P(A) = P(A \cap B^c) + P(A \cap B)$$

$$P(A \cap B^c) = P(A) - P(A \cap B)$$

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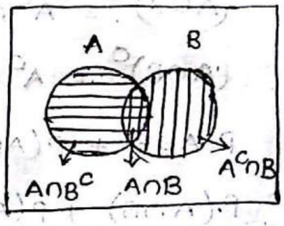
Theorem-5 :-

Addition theorem :- If A & B are any two events then $P(A\cup B) = P(A) + P(B) - P(A\cap B)$.



Proof :- Let A & B are any two events

Now $(A\cup B) = A\cup(A^c\cap B)$

Taking probability on b.s

$P(A\cup B) = P[A\cup(A^c\cap B)]$

Here A & $A^c\cap B$ are MEE

$P(A\cup B) = P(A) + P(A^c\cap B)$

From theorem ③

$P(A^c\cap B) = P(B) - P(A\cap B)$

$\therefore P(A\cup B) = P(A) + P(B) - P(A\cap B)$

Theorem-6 :-

Addition theorem for 3 events :- If A, B, C are any 3 events then

$P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(B\cap C) - P(C\cap A) + P(A\cap B\cap C)$

Proof :- A, B, C are any three events

Assume $A\cup B = D$

$P(A\cup B\cup C) = P(D\cup C)$

$= P(D) + P(C) - P(D\cap C)$

Replace $D = A\cup B$

$= P(A\cup B) + P(C) - P[(A\cup B)\cap C]$

$= P(A) + P(B) - P(A\cap B) + P(C) - P[(A\cap C)\cup(B\cap C)]$ (∵ By divisibility law)

$= P(A) + P(B) + P(C) - P(A\cap B) - [P(A\cap C) + P(B\cap C) - P[(A\cap C)\cap(B\cap C)]]$

$\therefore P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(A\cap C) - P(B\cap C) + P(A\cap B\cap C)$

Date 8/1/15

NOTE :-

1) If A & B are any two independent events, then

$P(A\cap B) = P(A)\cdot P(B)$

2) If A, B & C are any three independent events, then

$P(A\cap B\cap C) = P(A)\cdot P(B)\cdot P(C)$

4) If A, B are independent events, then A^c and B^c are also independent events.

$$P(A^c \cap B^c) = P(A^c) \cdot P(B^c)$$

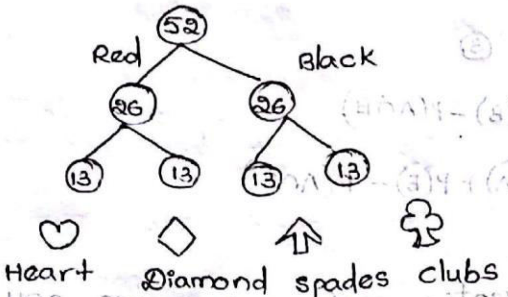
4) i) $(A \cup B)^c = A^c \cap B^c$

ii) $(A \cap B)^c = A^c \cup B^c$

iii) $P(A \cup B) + P(A \cup B)^c = 1$

iv) $P(A \cap B) + P(A \cap B)^c = 1$

5) Pack of cards is 52



13 are → A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, k, Q

Face cards are J, k, Q

Prob:-

1) The probabilities of three students to solve a problem in Mathematics are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ respectively. Find the probabilities that the problem to be solved in three members.

Sol:- Method - 1 :-

Given data,

$$P(A) = \frac{1}{2}, \quad P(B) = \frac{1}{3}, \quad P(C) = \frac{1}{4}$$

We have to find $P(A \cup B \cup C) = ?$

w.k.t $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

$= P(A) + P(B) + P(C) - P(A) \cdot P(B) - P(B) \cdot P(C) - P(C) \cdot P(A) + P(A) P(B) \cdot P(C)$

$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{6} - \frac{1}{12} - \frac{1}{8} + \frac{1}{24}$

$= \frac{12 + 8 + 6 - 4 - 2 - 3 + 1}{24}$

$= \frac{18}{24}$

$= \frac{3}{4}$

$= 0.75$

Method – 2 :-

w.k.t $\quad P(A\cup B\cup C) + P(A\cup B\cup C)^c = 1$

$$P(A\cup B\cup C) = 1 - P(A\cup B\cup C)^c$$
$$= 1 - P(A^c\cap B^c\cap C^c)$$
$$= 1 - \left[P(A^c)\cdot P(B^c)\cdot P(C^c)\right]$$
$$= 1 - \left[\frac{1}{\cancel{2}} \times \frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{3}}{4}\right]$$
$$= 1 - \frac{1}{4}$$
$$= \frac{3}{4} = 0.75$$

2) Two dice are thrown. Let 'A' be the event that the sum of points on the faces is 9. Let 'B' be the event that atleast one number is 6. Find i) $P(A\cup B)$ ii) $P(A\cap B)$ iii) $P(A^c\cup B^c)$

Sol:- Let A = sum of points on the faces is 9

$$A = \{(3,6)(6,3)(4,5)(5,4)\} = 4$$

$$P(A) = \frac{\text{Favourable events}}{\text{Total No. of events}}$$

$$P(A) = \frac{4}{36}$$

Let B = atleast one no. is 6 on two dice

$$B = \{(1,6)(2,6)(3,6)(4,6)(5,6)(6,1)(6,2)(6,3)(6,4)(6,5)(6,6)\}$$

$$= 11$$

$$P(B) = \frac{11}{36}$$

i) $P(A\cap B)$:-

$$A\cap B = \{(3,6)(6,3)\} = 2$$

$$P(A\cap B) = \frac{2}{36}$$

ii) $P(A \cup B)$:-

$A \cup B = \{(4,5)(5,4)(1,6)(2,6) \text{-----} (6,6)\}$

$$P(A \cup B) = \frac{13}{36}$$

iii) $P(A^c \cup B^c)$:-

W.k.t $P(A^c \cup B^c) = P(A \cap B)^c$

$$= 1 - P(A \cap B)$$

$$= 1 - \frac{2}{36}$$

$$\Rightarrow \frac{34}{36} = \frac{17}{18}$$

3) A class has 12 boys and 8 girls. Three students are selected at random one after another. Find the probability that i) first two are boys and third is girl ii) first and third are of same gender and the second is of opposite gender.

Sol:-

Total No. of students in a class = 20

No. of ways of selecting three students one after another is $n(S) = (20 \times 19 \times 18)$

i) Let E = first two are boys & third is girl

$= B \quad B \quad G$

$$P(E) = \frac{12 \times 11 \times 8}{20 \times 19 \times 18} = 0.154$$

ii) Let E = first and third are same gender and 2nd is opp. gender

$= B \quad G \quad B \quad \text{or} \quad G \quad B \quad G$

$\Rightarrow BGB + GBG$

$$= \frac{12 \times 8 \times 11}{20 \times 19 \times 8} + \frac{8 \times 12 \times 7}{20 \times 19 \times 8}$$

$= 0.1543 + 0.0982$

$= 0.2526$

4) Three students A, B, C are in running race. A and B have the same probability of winning and each is twice as likely win as 'C'. Find the probability that B or C wins.

Sol:- Given data,

$$P(A) = P(B) \text{ \& } P(A) = P(B) = 2P(C)$$

$P(B)$ or $P(C)$ to win = ?

$$P(B) + P(C) = ?$$

w.k.t $P(A) + P(B) + P(C) = 1$

$$2P(C) + 2P(C) + P(C) = 1$$

$$P(C) = \frac{1}{5}$$

$$P(B) = 2P(C)$$

$$= 2\left(\frac{1}{5}\right) = \frac{2}{5}$$

$$P(B) + P(C) = \frac{2}{5} + \frac{1}{5}$$

$$= \frac{3}{5}$$

5) **Conditional probability**:- If E_1 & E_2 are any two events in a sample space and $P(E_1) \neq 0$ then the probability of E_2 after the event E_1 has occured is called the conditional probability of the event E_2 given E_1 and it is denoted by $P\left(\frac{E_2}{E_1}\right)$, we define $P\left(\frac{E_2}{E_1}\right) = \frac{P(E_1 \cap E_2)}{P(E_1)}$

Similarly, $P\left(\frac{E_1}{E_2}\right) = \frac{P(E_1 \cap E_2)}{P(E_2)}$

Theorem :- General Multiplication theorem:-

Statement:- If A and B are any two events, then $P(A \cap B) = P(A) \cdot P\left(\frac{B}{A}\right) = P(B) \cdot P\left(\frac{A}{B}\right)$

Proof:-

Let A & B are any two events

By the definition of conditional probability, we have

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(B) \cdot P\left(\frac{A}{B}\right) \quad \text{--- (1)}$$

$$P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)}$$

$$P(A \cap B) = P(A) \cdot P\left(\frac{B}{A}\right) \quad \text{--- (2)}$$

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From ① & ②

$$P(A\cap B) = P(B)\cdot P\left(\frac{A}{B}\right) = P(A)\cdot P\left(\frac{B}{A}\right)$$

Multiplication theorem for 3 events:-

Statement:- prove that $P(A\cap B\cap C) = P(A)\cdot P\left(\frac{B}{A}\right)\cdot P\left(\frac{C}{A\cap B}\right)$

Proof:-

$$P[(A\cap B)\cap C] = P(A\cap B)\cdot P\left(\frac{C}{A\cap B}\right)$$

$$P(A\cap B\cap C) = P(A)\cdot P\left(\frac{B}{A}\right)\cdot P\left(\frac{C}{A\cap B}\right)$$

Theorem:- If E_1, E_2 are any two events in a sample space 'S' such that $E_1 \subseteq E_2$, then $P(E_1) \leq P(E_2)$

Proof:-

Given $E_1 \subseteq E_2 \Rightarrow E_2 - E_1 \geq 0$

$$E_2 \geq E_1$$

Taking Prob. on b.s

$$P(E_2) \geq P(E_1)$$

$$P(E_1) \leq P(E_2)$$

Independent event:- If two events are said as independent events, then $P\left(\frac{A}{B}\right) = P(A)$ & $P\left(\frac{B}{A}\right) = P(B)$

Theorem – Multiplication theorem for independent events:-

If A & B are independent events, then $P(A\cap B) = P(A)\cdot P(B)$

Proof:- If A & B are independent events, then

$$P\left(\frac{B}{A}\right) = P(B) \quad \& \quad P\left(\frac{A}{B}\right) = P(A)$$

By the statement of general multiplication theorem, we have

$$P(A\cap B) = P(A)\cdot P\left(\frac{B}{A}\right)$$

$$\therefore P(A\cap B) = P(A)\cdot P(B)$$

Prob.

1) If A & B are two events such that $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ & $P(A\cap B) = \frac{1}{5}$. then find i) $P(A\cup B)$ ii) $P(A\cap B^c)$ iii) $P(A^c\cap B)$ iv) $P(A^c\cap B^c)$

Sol:- Given data,

$$P(A) = \frac{1}{2},\ P(B) = \frac{1}{3},\ P(A\cap B) = \frac{1}{5}$$

i) $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{5}$$

$$= \frac{19}{30}$$

$$P(A\cup B) = 0.633$$

iii) $P(A^c \cap B) \Rightarrow$

$$-P(A\cap B) + P(B)$$

$$= -\frac{1}{5} + \frac{1}{3}$$

$$= 0.133$$

ii) $P(A\cap B^c) = P(A) - P(A\cap B)$

$$= \frac{1}{2} - \frac{1}{5}$$

$$= 0.3$$

iv) $P(A^c \cap B^c) = P((A\cup B)^c)$

$$= 1 - P(A\cup B)$$

$$= 1 - \frac{19}{30}$$

$$= 0.367$$

2) Two marbles are drawn in succession from a box containing 10 Red, 30 white, 20 Blue, 15 orange marbles with replacement being made after each draw. Find the probability that i) both are white ii) first is red, second is white.

Sol:- The no. of total marbles $= 10+30+20+15$

$= 75$

i) Let E_1 = event of the first marble is white

$$P(E_1) = \frac{30_{C_1}}{75_{C_1}} = \frac{30}{75}$$

Let E_2 = event of the second marble also is white (with replacement)

$$P\left(\frac{E_2}{E_1}\right) = \frac{30_{C_1}}{75_{C_1}} = \frac{30}{75}$$

The probability that both marbles are white $= P(E_1 \cap E_2)$

$$= P(E_1) \cdot P\left(\frac{E_2}{E_1}\right)$$

$$= \frac{30}{75} \times \frac{30}{75}$$

$$= \frac{4}{25}$$

$$= 0.16$$

ii) Let E_1 = event of the 1st marble is red

$$P(E_1) = \frac{10_{C_1}}{75_{C_1}} = \frac{10}{75}$$

Let E_2 = event of the second marble is white (with replacem

$$P\left(\frac{E_2}{E_1}\right) = \frac{30_{C_1}}{75_{C_1}} = \frac{30}{75}$$

The probability that the first marble is red & second marble is white $= P(E_1 \cap E_2)$

$$= P(E_1) \cdot P\left(\frac{E_2}{E_1}\right)$$

$$= \frac{10}{75} \times \frac{30}{75}$$

$$= 0.053$$

3) In a certain town 40% brown hair, 25% brown eyes, 15 of both brown hair & brown eyes. A person is selected at random. Find the probability that i) if he has brown hair, what is the probability that he has brown eyes also. ii) If he has brown eyes, determine the probability that he does not have brown hair.

Sol:- Given data,

$$P(H) = \frac{40}{100}$$

$$P(E) = \frac{25}{100}$$

$$P(H \cap E) = \frac{15}{100}$$

i) $$P\left(\frac{E}{H}\right) = \frac{P(E \cap H)}{P(H)} = \frac{15/100}{40/100}$$

$$= \frac{15}{40} = 0.375$$

ii) $P\left(\frac{H^c}{E}\right) = \frac{P(E\cap H^c)}{P(E)}$

$= \frac{P(E) - P(E\cap H)}{P(E)}$

$= \frac{\frac{25}{100} - \frac{15}{100}}{\frac{40}{100}}$

$= 0.4$

4) Three machines I, II, III produces 40%, 30%, 30% of the total no. of items of a factory. The percentage of defective items of these machines are 4%, 2%, 3%. If an item is selected at random, find the probability that the item is defective.

Sol:- Given data,

$P(A) = \frac{40}{100}$, $P(B) = \frac{30}{100}$, $P(C) = \frac{30}{100}$

$P\left(\frac{D}{A}\right) = \frac{4}{100}$, $P\left(\frac{D}{B}\right) = \frac{2}{100}$, $P\left(\frac{D}{C}\right) = \frac{3}{100}$

$P(D) = ?$

$P(D) = P(A\cap D)$ or $P(B\cap D)$ or $P(C\cap D)$

$= P(A\cap D) + P(B\cap D) + P(C\cap D)$

$= P(A)\cdot P\left(\frac{D}{A}\right) + P(B)\cdot P\left(\frac{D}{B}\right) + P(C)\cdot P\left(\frac{D}{C}\right)$

$= \frac{40}{100}\times\frac{4}{100} + \frac{30}{100}\times\frac{2}{100} + \frac{30}{100}\times\frac{3}{100}$

$P(D) = 0.031$

5) A and B are two events. Find the probability that i) neither A nor B occurs (iii) A occurs & B does not occurs iii) Exactly one of A and B occurs

Sol:- i) The probability that neither A nor B occurs $= P(A^c\cap B^c)$

$= P(A^c)\cdot P(B^c)$

$= [1-P(A)]\cdot[1-P(B)]$

$= 1 - P(B) - P(A) + P(A)\cdot P(B)$

$= 1 - P(A) - P(B) + P(A\cap B)$

$= 1 - [P(A) + P(B) - P(A\cap B)]$

$P(A^c\cap B^c) = 1 - P[A\cup B] = P(A\cup B)^c$

ii) The probability that A occurs & B does not occurs $= P(A \cap B^c)$

$$= P(A) \cdot P(B^c)$$
$$= P(A)\left[1 - P(B)\right]$$
$$= P(A) - P(A) \cdot P(B)$$
$$= P(A) - P(A \cap B)$$
$$= P(A \cap B^c)$$

iii) The probability that exactly one of A & B occurs

$$= P(A \cap B^c) + P(A^c \cap B)$$
$$= P(A) \cdot P(B^c) + P(A^c) \cdot P(B)$$
$$= P(A)\left[1 - P(B)\right] + \left[1 - P(A)\right] \cdot P(B)$$
$$= P(A) - P(A) \cdot P(B) + P(B) - P(A) P(B)$$
$$= P(A) + P(B) - 2P(A \cap B)$$
$$= P(A \cup B) - P(A \cap B)$$

Date
10/1/15

Theorem:- If A, B and C are mutually exclusive independent event then prove that $A \cup B$ & C are also independent events.

Proof:-

Given A, B and C are mutually exclusive & independent events i.e,
$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$$

We have to prove that $(A \cup B)$ & C are independent events

i.e., $P[(A \cup B) \cap C] = P(A \cup B) \cdot P(C)$

LHS:- $P[(A \cup B) \cap C] = P[(A \cap C) \cup (B \cap C)]$

$$= P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$$
$$= P(A) \cdot P(C) + P(B) \cdot P(C) - P(A) \cdot P(B) \cdot P(C)$$
$$= P(C)\left[P(A) + P(B) - P(A) \cdot P(B)\right]$$
$$= P(C)\left[P(A \cup B)\right] - \text{RHS}.$$

Theorem:- A and B are independent events. Then prove that A & B' are also independent events.

Proof:- Given A & B are independent events

we have to prove that A & B^c are independent events

i.e., $P(A\cap B^c) = P(A)\cdot P(B^c)$

LHS:- $P(A\cap B^c) = P(A) - P(A\cap B)$

$= P(A) - P(A)\cdot P(B)$

$= P(A)\cdot[1 - P(B)]$

$= P(A)\cdot P(B^c)$ — RHS.

Theorem:- A, B, C are events in a sample space 'S'. If A is independent of events B, $B\cup C$ & $B\cap C$ then prove that A, C are independent events.

Proof:- Let A is independent of events B, $B\cup C$, $B\cap C$

i.e., $P(A\cap B) = P(A)\cdot P(B)$ —— ①

$P[A\cap(B\cup C)] = P(A)\cdot P(B\cup C)$ —— ②

$P[A\cap(B\cap C)] = P(A)\cdot P(B\cap C)$ —— ③

we have to prove that $P(A\cap C) = P(A)\cdot P(C)$

From ②,

$P[(A\cap B)\cup(A\cap C)] = P(A)[P(B) + P(C) - P(B\cap C)]$

$P(A\cancel{\cap B}) + P(A\cap C) - P(A\cancel{\cap B}\cap C) = P(A)\cdot P(B) + P(A)\cdot P(C) - P(A)\cdot P(B\cap C)$

$= \cancel{P(A\cap B)} + P(A)\cdot P(C) - P(A)\cdot \cancel{P(B)}\cdot P(C)$

$P(A\cap C) = P(A)\cdot P(C)$

Dec 2013

Prob:-

1) a) If A & B are mutually exclusive events, $P(A) = 4P(B)$ & $A\cup B = S$.

Find i) $P(A^c\cap B)$ ii) $P(A^c\cap B^c)$

Sol:- Given $P(A) = 4P(B)$

$A\cup B = S$

Taking probability on b/s

$P(A\cup B) = P(S)$

$P(A) + P(B) = 1$

$4P(B) + P(B) = 1$

$5P(B) = 1$

$P(B) = \frac{1}{5}$, $P(A) = \frac{4}{5}$

i) $P(A^c \cap B) = P(B) - P(A \cap B)$

$= P(B) - P(A) \cdot P(B)$

$= \frac{1}{5} - 0$

$P(A^c \cap B) = \frac{1}{5}$

ii) $P(A^c \cap B^c) = P(A \cup B)^c$

$= 1 - P(A \cup B)$

$\Rightarrow 1 - 1 = 0$

(or)

$P(A^c) \cdot P(B^c)$

$= \frac{1}{5} \times \frac{4}{5}$

$= \frac{4}{25}$

b). If $P(A) = \frac{2}{3}$ and $P(B) = \frac{1}{5}$, P.T $\frac{2}{15} \leq P(A \cap B) \leq \frac{1}{5}$

Sol:-

Given $P(A) = \frac{2}{3}$, $P(B) = \frac{1}{5}$

we know that

$A \cap B \subseteq A \cap B$

Taking probability on b.s

$P(A \cap B) \leq P(A \cap B)$

Here $P(A) \cdot P(B) \leq P(A \cap B)$

$\left(\frac{2}{3}\right)\left(\frac{1}{5}\right) \leq P(A \cap B)$

$\frac{2}{15} \leq P(A \cap B)$ —— ①

we know that $A \cap B \subseteq B$

Taking probability on both sides

$P(A \cap B) \leq P(B)$

$P(A \cap B) \leq \frac{1}{5}$ —— ②

From ① & ②

$\frac{2}{15} \leq P(A \cap B) \leq \frac{1}{5}$

20/1/15

Baye's theorem :-

Statement :- If events $E_1, E_2 - - - - E_n$ are mutually exclusive events, and A is any arbitrary event then

$$P\left(\frac{E_i}{A}\right) = \frac{P\left(\frac{A}{E_i}\right) \cdot P(E_i)}{\sum_{i=1}^{n} P\left(\frac{A}{E_i}\right) \cdot P(E_i)}$$

Proof :-

Let $E_1, E_2, - - - - - E_n$ are mutually exclusive events and A is any arbitrary event in a sample space S

$$\Rightarrow S = E_1 \cup E_2 \cup E_3 \cup - - - - - \cup E_n$$

Now $A \cap S = A$

$$A = A \cap \{E_1 \cup E_2 \cup E_3 \cup - - - \cup E_n\}$$

$$= (A \cap E_1) \cup (A \cap E_2) - - - - - \cup (A \cap E_n)$$

Taking probability on both sides,

$$P(A) = P[(A \cap E_1) \cup (A \cap E_2) - - - - - \cup (A \cap E_n)]$$

Here $A \cap E_1, A \cap E_2 - - - - - - A \cap E_n$ are Mutually exclusive events

$$P(A) = P(A \cap E_1) + P(A \cap E_2) + - - - - + P(A \cap E_n)$$

$$P(A) = \sum_{i=1}^{n} P(A \cap E_i) \quad \text{——} \; ①$$

By the definition of conditional probability, we have

$$P\left(\frac{E_i}{A}\right) = \frac{P(A \cap E_i)}{P(A)} \quad ②$$

From ① in ②, $$P\left(\frac{E_i}{A}\right) = \frac{P(A \cap E_i)}{\sum_{i=1}^{n} P(A \cap E_i)} \quad \text{——} \; ③$$

Now $$P\left(\frac{A}{E_i}\right) = \frac{P(A \cap E_i)}{P(E_i)}$$

$$P(A \cap E_i) = P(E_i) \cdot P\left(\frac{A}{E_i}\right) \text{——} \; ④$$

Taking summation on b.s from $i = 1$ to n

$$\sum_{i=1}^{n} P(A \cap E_i) = \sum_{i=1}^{n} P(E_i) \cdot P\left(\frac{A}{E_i}\right) \text{——} \; ⑤$$

Sub. ④, ⑤ in ③ we get

From ③ $\Rightarrow P\left(\frac{E_i}{A}\right) = \dfrac{P(E_i) \cdot P\left(\frac{A}{E_i}\right)}{\sum_{i=1}^{n} P(E_i) \cdot P\left(\frac{A}{E_i}\right)}$

Hence the theorem is proved.

Prob:-

1) In a bolt factory, machines A, B & C manufacturer 20%, 30%, 50% of the total of their output and 6%, 3% & 2% are defective. A bolt is drawn at random and found to be defective. Find the probability that manufactured from i) machine A ii) machine B iii) machine C.

Sol:- Given, $P(A) = 20\% = 0.2$

$P(B) = 30\% = 0.3$

$P(C) = 50\% = 0.5$

$P(D/A) = 6\% = 0.06$

$P(D/B) = 3\% = 0.03$

$P(D/C) = 2\% = 0.02$

i) Machine A:-

$P\left(\frac{A}{D}\right) = ?$

By the statement of Baye's theorem, we have

$$P\left(\frac{A}{D}\right) = \frac{P(A) \cdot P\left(\frac{D}{A}\right)}{P(D/A) \cdot P(A) + P(D/B) \cdot P(B) + P(D/C) \cdot P(C)}$$

$$= \frac{(0.2)(0.06)}{(0.06)(0.2)+(0.03)(0.3)+(0.02)(0.5)}$$

$$P\left(\frac{A}{D}\right) = \frac{12}{31}$$

ii) Machine B:-

$P\left(\frac{B}{D}\right) = ?$

$$P\left(\frac{B}{D}\right) = \frac{P(B) \cdot P\left(\frac{D}{B}\right)}{P(D/A) \cdot P(A) + P(D/B) \cdot P(B) + P(D/C) \cdot P(C)}$$

$$= \frac{(0.3)(0.03)}{(0.06\times0.2)+(0.03\times0.3)+(0.02\times0.5)}$$

$$= \frac{9}{31}.$$

iii) <u>Machine C</u>:-

$$P\left(\frac{C}{D}\right) = \frac{(0.5\times0.02)}{(0.06\times0.2)+(0.03\times0.3)+(0.02\times0.5)}$$

$$= \frac{10}{31}$$

2) Suppose 5 men out of 100 and 25 women out of 10,000 are Colour blind. A colour blind person is selected at random. what is the probability of the person being a male (Assume male & female to be equal numbers)

<u>Sol</u>:- Given $P(B/M) = \frac{5}{100}$

$$P(B/W) = \frac{25}{10,000}$$

To find $P\left(\frac{M}{B}\right) = ?$

$$P(M) = P(W) = \frac{1}{2}$$

By the statement of Baye's theorem, we have

$$P\left(\frac{M}{B}\right) = \frac{P\left(\frac{B}{M}\right).P(M)}{P\left(\frac{B}{M}\right).P(M)+P\left(\frac{B}{W}\right).P(W)}$$

$$= \frac{\frac{5}{100}\times\frac{1}{2}}{\frac{5}{100}\times\frac{1}{2}+\frac{25}{10,000}\times\frac{1}{2}}$$

$$P\left(\frac{M}{B}\right) = \frac{20}{21}$$

3) A bag contains 2 white & 3 red balls and another bag contains 4 white & 5 red balls. One ball is drawn at random from one of the bag and it is found to be red. Find the probability that the red ball drawn is from bag B.

Sol:- Given $P\left(\frac{W}{A}\right) = \frac{2}{5}$

$$P\left(\frac{R}{A}\right) = \frac{3}{5}$$

$$P\left(\frac{W}{B}\right) = \frac{4}{9}, \quad P\left(\frac{R}{B}\right) = \frac{5}{9}$$

$$P\left(\frac{B}{R}\right) = ?$$

$$P(A) = P(B) = \frac{1}{2}$$

By the statement of Baye's theorem, we have

$$P\left(\frac{B}{R}\right) = \frac{P\left(\frac{R}{B}\right) \cdot P(B)}{P\left(\frac{R}{A}\right) \cdot P(A) + P\left(\frac{R}{B}\right) \cdot P(B)}$$

$$= \frac{\frac{5}{9} \times \frac{1}{2}}{\frac{3}{5} \times \frac{1}{2} + \frac{5}{9} \times \frac{1}{2}}$$

$$P\left(\frac{B}{R}\right) = \frac{25}{52}$$

4) A business man goes to hotels (X, Y, Z. 20%, 50%, 30% of the time respectively. It is known that 5%, 4%, 8% of the rooms in X, Y, Z hotels have faulty plumbings. What is the probability that the business man's room having faulty plumbing is assigned to hotel Z.

Sol:-

$$P\left(\frac{F}{X}\right) = 5\%, \quad P\left(\frac{F}{Y}\right) = 4\%, \quad P\left(\frac{F}{Z}\right) = 8\%$$

$$P(X) = 20\%, \quad P(Y) = 50\%, \quad P(Z) = 30\%$$

$$P\left(\frac{Z}{F}\right) = ?$$

$$P\left(\frac{Z}{F}\right) = \frac{P\left(\frac{F}{Z}\right) \times P(Z)}{P\left(\frac{F}{X}\right) \times P(X) + P\left(\frac{F}{Y}\right) P(Y) + P\left(\frac{F}{Z}\right) P(Z)}$$

$$= \frac{(0.08)(0.3)}{(0.08 \times 0.3) + ((0.05) \times 0.2) + (0.04 \times 0.5)}$$

$$P\left(\frac{Z}{F}\right) = \frac{4}{9}$$

Date 1/1/15

5) In a factory machine A produces 40% of the output and machine B produces 60% of the output and in average 9 items in 1000 produced by A are defective, 1 item in 250 items produced by B is defective. An item is drawn at random from a

day's output is defective. what is the probability that it was found by A or B.

Sol:- Given $P(A) = 40\%$.

$P(B) = 60\%$.

$P(D/A) = \frac{9}{1000}$

$P\left(\frac{D}{B}\right) = \frac{1}{250}$

$P\left(\frac{A}{D}\right) + P\left(\frac{B}{D}\right) = ?$

$$P\left(\frac{A}{D}\right) = \frac{P\left(\frac{D}{A}\right) \cdot P(A)}{P\left(\frac{D}{A}\right) \cdot P(A) + P\left(\frac{D}{B}\right) \cdot P(B)} = 0.6$$

$$P\left(\frac{B}{D}\right) = \frac{P\left(\frac{D}{B}\right) \cdot P(B)}{P\left(\frac{D}{B}\right) \cdot P(B) + P\left(\frac{D}{A}\right) \cdot P(A)} = 0.4$$

Random variables:- A random variable 'X' whose value is determined by the outcome of random experiment is called a random variable

Ex-1:- Tossing two coins, no. of heads are $X = \{0, 1, 2\}$

Ex-2:- Tossing three coins, no. of heads are $X = \{0, 1, 2, 3\}$

Random variables are two types.

1. Discrete Random variable
2. Continuous Random variable

1. **Discrete Random variable**:- A random variable X which can take only a finite no. of discrete values in the interval of domain is called a discrete random variable.

In other words, if the random variable takes the values only on the set $\{0, 1, 2 ------ n\}$ is called a discrete random variable.

Ex:- The random variable denoting the no. of students in a class is $X(n) = \{x, x$ is a +ve integer$\}$

2. **Continuous Random variable**:- A random variable X which can take values continuously.

Ex:- height, weight.

Probability distribution function:- If X is a random variable then the function $f(x_i) = P(X = x_i) = P(x_i)$ is called probability distribution function. There are two types of probability distribution.

1. Discrete probability distribution function or probability mass function (pmf)
2. Continuous probability distribution function or probability density function (pdf)

1. **P.m.f :-** Suppose 'X' is a discrete random variable with possible outcomes $x_1, x_2 ------ x_n$, the probability of each possible x_i is $P(X = x_i) = P(x_i)$ for $i = 1, 2 --- n$. If the numbers $P(x_i)$ satisfy the following two conditions,

i) $P(x_i) \geq 0 \ \forall i$

ii) $\sum_{i=1}^{n} P(x_i) = 1$ (or) $\sum P(x_i) = 1$

2. **P.d.f :-** If X is a continuous random variable, then the probability distribution $P(X = x_i) = P(x_i)$ is said to be continuous probability distribution, If it is following two properties

i) $P(x_i) \geq 0 \ \forall i$

ii) $\int_{-\infty}^{\infty} P(x_i)\, dx = 1$

Mean, variance & Standard deviation of Pmf:-

1) **Mean (μ):-**

$$\mu = E(x) = \sum_{i=1}^{n} x_i P(x_i) \quad \{E = \text{Expectation}\}$$

2) **Variance (σ^2):-**

$$\sigma^2 = \sum_{i=1}^{n} x_i^2 P(x_i) - \left[\sum_{i=1}^{n} x_i P(x_i)\right]^2$$

$$= E(x^2) - [E(x)]^2$$

3) **Standard deviation (σ):-**

$$\sigma = \sqrt{\sum_{i=1}^{n} x_i^2 P(x_i) - \sum_{i=1}^{n} x_i P(x_i)}$$

$$\sigma = \sqrt{E(x^2) - [E(x)]^2}$$

Mean, variance & standard deviation of Pdf:-

1) Mean (μ) :-

$$\mu = E(x) = \int_{-\infty}^{\infty} x_i \, p(x_i) \, dx$$

2) Variance (σ^2):-

$$\sigma^2 = \int_{-\infty}^{\infty} x_i^2 \cdot p(x_i) \, dx - \left[\int_{-\infty}^{\infty} x_i \, p(x_i) \, dx \right]^2$$

$$= E(x^2) - [E(x)]^2$$

3) Standard deviation (σ):-

$$\sigma = \sqrt{\int_{-\infty}^{\infty} x_i^2 p(x_i) \, dx - \left[\int_{-\infty}^{\infty} x_i p(x_i) \, dx \right]^2}$$

$$= \sqrt{E(x^2) - [E(x)]^2}$$

Properties of Mean & variance :-

Mean

1) $E(a) = a$
2) $E(ax) = a \cdot E(x)$
3) *** $E(ax+b) = a \cdot E(x) + b$
4) $E(x+y) = E(x) + E(y)$
5) $E(xy) = E(x) \cdot E(y)$
6) $E(ax+by) = a \cdot E(x) + b\,E(y)$

Variance

1) $V(a) = 0$
2) *** $V(ax) = a^2 \cdot V(x)$
3) *** $V(ax+b) = a^2 V(x)$
4) $V(x+y) = V(x) + V(y)$
5) $V(xy) = V(x) \cdot V(y)$
6) $V(ax+by) = a^2 V(x) + b^2 V(y)$

If $F(x)$ is given, then $f(x) = \frac{d}{dx}[F(x)]$

Prob:-

1) A random variable X has the following probability function.

X	0	1	2	3	4	5	6	7
$P(X=x_i)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

i) find k ii) Evaluate $P(X<6)$, $P(X\geq 6)$, $P(0<x<5)$ and $P(0\leq X \leq 4)$ iii) If $P(X\leq K) > \frac{1}{2}$, find the minimum value of K. iv) Mean v) variance.

By the definition of pmf, we have

$$\Sigma P(x_i) = 1 \quad (\text{or}) \quad \sum_{i=0}^{7} P(x_i) = 1$$

$$P(0+k+2k+2k+3k+k^2+2k^2+7k^2+k) = 1$$

$$10k^2+9k = 1$$

$$10k^2+9k-1 = 0$$

$$10k^2+10k-k-1 = 0$$

$$10k(k+1)-1(k+1) = 0$$

$$\cancel{k=-1}, \quad k = 0.1$$

$$\therefore k = 0.1$$

$P(x<6)$:-

Method-I:-

$$P(x<6) = P(x=0)+P(x=1)+P(x=2)+P(x=3)+P(x=4)+P(x=5)+P(x=6)$$

$$= 0+k+2k+2k+3k+k^2$$

$$= k^2+8k$$

$$= 8(0.1)+(0.1)^2$$

$$= (0.8+0.01)$$

$$= 0.81$$

Method-II:-

$$P(x<6) = 1-P(x\geq 6)$$

$$= 1-[P(x=6)+P(x=7)]$$

$$= 1-2k^2-7k^2-k$$

$$= -9k^2-k+1$$

$$= -9(0.1)^2-(0.1)+1$$

$$= 0.81$$

$\underline{\underline{P(x \geq 6)}}$:-

$$P(x \geq 6) = P(x=6) + P(x=7)$$
$$= 2k^2 + 7k^2 + k$$
$$= 9k^2 + k$$
$$= 0.19$$

$\underline{\underline{P(0 < x < 5)}}$:-

$$P(0 < x < 5) = P(x=1) + P(x=2) + P(x=3) + P(x=4)$$
$$= k + 2k + 2k + 3k$$
$$= 8k$$
$$= 0.8$$

$\underline{\underline{P(0 \leq x \leq 4)}}$:-

$$P(0 \leq x \leq 4) = P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4)$$
$$= 0 + k + 2k + 2k + 3k^2$$
$$= 3k^2 + 5k$$
$$= 8k$$
$$= 0.8$$

iii) $P(x \leq k) > \frac{1}{2} \Rightarrow k = 0 \Rightarrow P(x \leq 0) > \frac{1}{2}$

$$P(x=0) > \frac{1}{2}$$
$$0 > \frac{1}{2}$$

Not satisfied

$$k = 1 \Rightarrow P(x \leq 1) > \frac{1}{2}$$
$$P(x=0) + P(x=1) > \frac{1}{2}$$
$$0 + k > \frac{1}{2}$$
$$0.1 > \frac{1}{2}$$

Not satisfied

$$k = 2 \Rightarrow P(x \leq 2) > \frac{1}{2}$$
$$P(x=0) + P(x=1) + P(x=2) > \frac{1}{2}$$
$$0 + k + 2k > \frac{1}{2}$$
$$0.3 > \frac{1}{2}$$

Not satisfied

$$k = 3 \Rightarrow P(x \leq 3) > \frac{1}{2}$$
$$P(x=0) + P(x=1) + P(x=2) + P(x=3) > \frac{1}{2}$$

$$0+k+2k+2k > \frac{1}{2}$$

$$5k > \frac{1}{2}$$

$$0.5 > \frac{1}{2}$$

Not satisfied

$$k=4 \Rightarrow P(x \leq 4) > \frac{1}{2}$$

$$P(x=0)+P(x=1)+P(x=2)+P(x=3)+P(x=4) > \frac{1}{2}$$

$$0+k+2k+2k+3k > \frac{1}{2}$$

$$0.8 > \frac{1}{2}$$

Satisfied

$$\therefore k=4$$

iv) **Mean:-**

$$\mu = E(x) = \sum_{i=0}^{7} x_i P(x_i)$$

$$= 0+1(k)+2(2k)+3(2k)+4(3k)+5(k^2)+6(2k^2)+7(7k^2+k)$$

$$= k+4k+6k+12k+5k^2+12k^2+49k^2+7k$$

$$= 66k^2+30k$$

$$\mu = 3.66$$

v) **Variance:-**

$$\sigma^2 = \sum_{i=0}^{7} x_i^2 P(x_i) - \left[\sum_{i=0}^{7} x_i P(x_i)\right]^2$$

$$\sigma^2 = \left[0(0)+1(k)+4(2k)+9(2k)+16(3k)+25(k^2)+36(2k^2)+49(7k^2+k)\right] - [3.66]^2$$

$$= \left[k+8k+18k+48k+25k^2+72k^2+343k^2+49k\right] - (3.66)^2$$

$$= (440k^2+124k) - (3.66)^2$$

$$\sigma^2 = 3.4044$$

2) A random variable X is different as the sum of the numbers on the faces when two dice are thrown. Find the mean of X.

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Sol:- The probability distribution function is

X	2	3	4	5	6	7	8	9	10	11	12
$P(X=x_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Mean $\mu = \sum_{i=2}^{12} x_i P(x_i)$

$$= 2\left(\frac{1}{36}\right)+3\left(\frac{2}{36}\right)+4\left(\frac{3}{36}\right)+5\left(\frac{4}{36}\right)+6\left(\frac{5}{36}\right)+7\left(\frac{6}{36}\right)+8\left(\frac{5}{36}\right)+9\left(\frac{4}{36}\right)+10\left(\frac{3}{36}\right)+11\left(\frac{2}{36}\right)+12\left(\frac{1}{36}\right)$$

$\mu = 7$

3) A sample of 4 items is selected at random from a box containing 12 items of which 5 are defective. Find the expected no. of defective items

Sol:- Let X denote the no. of defective items

$X = \{0,1,2,3,4\}$

$\left({}^nC_r = \frac{n!}{(n-r)!\,r!} \qquad {}^nP_r = \frac{n!}{(n-r)!} \right)$

$$P(X=0) = \frac{{}^7C_4}{{}^{12}C_4} = \frac{35}{495}$$

$$P(X=1) = \frac{{}^7C_3 \times {}^5C_1}{{}^{12}C_4} = \frac{175}{495}$$

$$P(X=2) = \frac{{}^7C_2 \times {}^5C_2}{{}^{12}C_4} = \frac{210}{495}$$

$$P(X=3) = \frac{{}^7C_1 \times {}^5C_3}{{}^{12}C_4} = \frac{70}{495}$$

$$P(X=4) = \frac{{}^7C_0 \times {}^5C_4}{{}^{12}C_4} = \frac{5}{495}$$

The probability distribution

X	0	1	2	3	4
$P(x_i)$	$35/495$	$175/495$	$\frac{210}{495}$	$\frac{70}{495}$	$\frac{5}{495}$

Mean $\mu = \sum_{i=0}^{4} x_i P(x_i)$

$= 35 + 175 + 12(210) + 12(10) + 1(5)/_{495}$

$\mu = \frac{5}{3} = 1.66$

4) Given that $f(x) = \frac{k}{2x}$ is a probability distribution for a random variable $x = 1, 2, 3, 4$. Find i) k ii) Mean iii) Variance

Sol:- The probability distribution is

x_i	1	2	3	4
$P(x_i)$	$\frac{k}{2}$	$\frac{k}{4}$	$\frac{k}{6}$	$\frac{k}{8}$

i) Sum of probabilities = 1

$$\frac{k}{2} + \frac{k}{4} + \frac{k}{6} + \frac{k}{8} = 1$$

$$\frac{25}{24} k = 1$$

$$k = \frac{24}{25} = 0.96$$

ii) Mean $\mu = \sum_{i=1}^{4} x_i P(x_i)$

$$\mu = \frac{k}{2} + \frac{2k}{4} + \frac{3k}{6} + \frac{4k}{8}$$

$$= \frac{4k}{2}$$

$$\mu = 1.92$$

iii) Variance $\sigma^2 = \sum_{i=1}^{4} x_i^2 P(x_i) - \left[\sum_{i=1}^{4} x_i P(x_i)\right]^2$

$$= \left[\frac{k}{2} + k + \frac{9k}{6} + \frac{16k}{8}\right] - \left[1.92\right]^2$$

$$= 5k - (1.92)^2$$

$$\sigma^2 = 1.1136$$

5) Find Mean and variance of the probability distribution given by $f(x) = \frac{1}{n}$, for $x = 1, 2, ---n$

Sol:- The probability distribution

x	1	2	3	...	n
$f(x)=\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	...	$\frac{1}{n}$

i) Mean $\mu = \sum_{i=1}^{n} x_i f(x_i)$

$= 1\left(\frac{1}{n}\right)+2\left(\frac{1}{n}\right)+3\left(\frac{1}{n}\right)+ \cdots\cdots + n\left(\frac{1}{n}\right)$

$= \frac{1}{n}+\frac{2}{n}+\frac{3}{n}+\cdots\cdots+\frac{n}{n}$

$= \frac{1}{n}\left[1+2+\cdots\cdots+n\right]$

$= \frac{1}{n}\left[\frac{n(n+1)}{2}\right]$

$= \frac{n+1}{2}$

ii) variance $\sigma^2 = \sum_{i=1}^{n} x_i^2 f(x_i) - \left[\sum_{i=1}^{n} x_i f(x_i)\right]^2$

$=\left[1^2\left(\frac{1}{n}\right)+2^2\left(\frac{1}{n}\right)+3^2\left(\frac{1}{n}\right)+\cdots\cdots+n^2\left(\frac{1}{n}\right)\right]-\mu^2$

$= \frac{1}{n}\left[1^2+2^2+3^2+\cdots\cdots+n^2\right]-\mu^2$

$= \frac{1}{n}\left[\frac{n(n+1)(2n+1)}{6}\right]-\mu^2$

$= \frac{2n^2+3n+1}{6} - \frac{(n+1)^2}{4}$

$= \frac{4n^2+6n+2-3(n+1)^2}{12}$

$= \frac{4n^2+\cancel{6n}+2-3n^2-3-\cancel{6n}}{12}$

$= \frac{n^2-1}{12}$

6) A continuous random variable X has the distribution function $F(x) = \begin{cases} 0 & \text{if } x \le 1 \\ K(x-1)^4 & \text{if } 1 < x \le 3 \\ 1 & \text{if } x > 3 \end{cases}$

find i) $f(x)$ ii) K iii) Mean iv) Variance.

Sol:- i) we know that $f(x) = \frac{d}{dx} F(x)$

$$f(x) = \begin{cases} 0 & \text{if } x \le 1 \\ 4k(x-1)^3 & \text{if } 1 < x \le 3 \\ 0 & \text{if } x > 3 \end{cases}$$

~~ii)~~ (Sum ~~of~~ probabilities = 1

$\cancel{k((x-1)^4+1) = 1}$

$k(x-1)^4 = 0$

$\cancel{k = -(x-1)^4}$)

ii) W.k.t $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\int_{-\infty}^{1} f(x)\, dx + \int_{1}^{3} f(x)\, dx + \int_{3}^{\infty} f(x)\, dx = 1$$

$$\int_{-\infty}^{1} 0\, dx + \int_{1}^{3} 4k(x-1)^3\, dx + \int_{3}^{\infty} 0\, dx = 1$$

$$0 + 4k\int_{1}^{3} (x-1)^3\, dx + 0 = 1$$

$$4k\left[\frac{(x-1)^4}{4}\right]_1^3 = 1$$

$$4k\left[\frac{1}{4}(2^4 - 0)\right] = 1$$

$$4k\left[\frac{\cancel{16}^{\,4}}{\cancel{4}}\right] = 1$$

$$k = \frac{1}{16}$$

$$k = 0.0625$$

iii)
$$\mu = \int_{-\infty}^{\infty} x\, p(x_i)\, dx$$

$$\mu = \int_{-\infty}^{1} x f(x)\, dx + \int_{1}^{3} x \cdot f(x)\, dx + \int_{3}^{\infty} x f(x)\, dx$$

$$= \int_{-\infty}^{1} 0\, dx + \int_{1}^{3} x \cdot 4k(x-1)^3\, dx + \int_{3}^{\infty} 0\, dx$$

$$= 4k\int_{1}^{3} x(x-1)^3\, dx$$

$$= 4k\left[x \times \frac{(x-1)^4}{4} - 1\,\frac{(x-1)^5}{4(5)} + 0\right]_1^3 \qquad \left[\because \int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots\right]$$

$$= 4k\left[\left(\frac{3\times 2^4}{4} - \frac{2^5}{20}\right) - 0\right]$$

$$= 4(0.0625)\left[\frac{3\times \cancel{16}^{4}}{\cancel{4}} - \frac{32}{20}\right]$$

$$\mu = 2.6$$

iv)

$$V(x) = \sigma^2 = \int_{-\infty}^{\infty} x^2 p(x)\,dx - \mu^2$$

$$\sigma^2 = \left[\int_{-\infty}^{1} x^2 f(x)\,dx + \int_{1}^{3} x^2\cdot f(x)\,dx + \int_{3}^{\infty} x^2 f(x)\,dx\right] - \mu^2$$

$$= \left[\int_{-\infty}^{1} 0\,dx + \int_{1}^{3} x^2\cdot 4k(x-1)^3\,dx + \int_{3}^{\infty} 0\,dx\right] - \mu^2$$

$$= \left[4k\int_{1}^{3} x^2 (x-1)^3\,dx\right] - \mu^2$$

$$= 4k\left[x^2\,\frac{(x-1)^4}{4} - \frac{2x\,(x-1)^5}{5\times 4} + \frac{2(x-1)^6}{120}\right]_1^3 - \mu^2$$

$$= 4(0.0625)\left[\frac{9\times \cancel{16}^{4}}{\cancel{4}} - \frac{6(\cancel{32}^{8})}{\cancel{20}_{5}} + \frac{2(64)}{120}\right] - (2.6)^2$$

$$\sigma^2 = 0.106$$

7) The cumulative distribution function for a continuous random variable

$$F(x) = \begin{cases} 1-e^{-2x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Find i) $f(x)$ ii) Mean iii) variance.

Sol:-

$$f(x) = \frac{d}{dx} F(x)$$

i) $$f(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

ii) Mean $$\mu = \int_{-\infty}^{\infty} x\cdot f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\, dx + 2\int_{0}^{\infty} x\cdot e^{-2x}\, dx$$

$$= 2\left[x\cdot \frac{e^{-2x}}{-2} - 1\cdot\frac{e^{-2x}}{4}\right]_0^{\infty}$$

$$= 2\left[(0-0) - \left(0-\frac{1}{4}\right)\right]$$

$$= \left(\frac{1}{4}\right)\times 2$$

$$\mu = \frac{1}{2}$$

iii)

$$\sigma^2 = \int_0^{\infty} x^2\cdot f(x)\, dx - \mu^2$$

$$= 2\int_0^{\infty} x^2\cdot e^{-2x}\, dx - (0.5)^2$$

$$= 2\left[x^2\cdot\frac{e^{-2x}}{-2} - 2x\cdot\frac{e^{-2x}}{-4} + 2\cdot\frac{e^{-2x}}{-8}\right]_0^{\infty} - (0.5)^2$$

$$= 2\left[(0-0-0) - \left(0-0+2\left(-\frac{1}{8}\right)\right)\right] - (0.5)^2$$

$$= 0.25$$

Date 23/1/15

8) Let x denote the minimum of the two numbers that appear when a pair of fair dice is thrown once. Find

a) Discrete probability distribution b) Mean c) variance.

Sol:- when two dice are thrown, then the total no. of outcomes $= 6\times 6$

$= 36$ i.e.,

$$\left\{\begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{matrix}\right\} = 36$$

If the random variable x is assigned the min. of two numbers then sample space

$$S = \left\{\begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ \cancel{1} & \cancel{2} & \cancel{3} & \cancel{4} & \cancel{5} & \end{matrix}\right\}$$

$\Downarrow$ 36

$x = \{1, 2, 3, 4, 5, 6\}$

i) The probability distribution is

x_i	1	2	3	4	5	6
$P(x_i)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

b) Mean $\mu = \sum_{i=1}^{6} x_i \cdot P(x_i)$

$$\mu = 1\left(\frac{11}{36}\right) + 2\left(\frac{9}{36}\right) + 3\left(\frac{7}{36}\right) + 4\left(\frac{5}{36}\right) + 5\left(\frac{3}{36}\right) + 6\left(\frac{1}{36}\right)$$

$$= 2.52$$

c) variance $\sigma^2 = \sum_{i=1}^{6} x_i^2 \cdot P(x_i) - \mu^2$

$$= 8.361 - (2.52)^2$$

$$\sigma^2 = 2.01$$

2013-DEC Imp

9) The probability density function $f(x)$ is a continuous random variable. $f(x) = ce^{-|x|}$, $-\infty < x < \infty$. show that $c = \frac{1}{2}$ and find that mean and variance of the distribution and also find the probability that the variate lies b/w 0 and 4.

Sol:- Given $f(x) = ce^{-|x|}$, $-\infty < x < \infty$

i) To find c:- We know that $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{\infty} ce^{-|x|}\,dx = 1 \quad \{\because f(-x) = f(x)\}$$

Here $f(x)$ is even

$$2\int_{0}^{\infty} ce^{-|x|}\,dx = 1$$

$$2c\int_{0}^{\infty} e^{-x}\,dx = 1$$

18

$$2c\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1$$

$$2c\left[-e^{-\infty}-(-e^{-0})\right]=1$$

$$2c\left[0+1\right]=1$$

$$c=\frac{1}{2}$$

ii) Mean $\mu = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$

$$= \int_{-\infty}^{\infty} x \cdot ce^{-|x|}\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} x \cdot e^{-|x|}\,dx$$

⇒ Here above function is odd,

then $\int_{-\infty}^{\infty} f(x)\,dx = 0$

$$= \frac{1}{2}(0)$$

$$\mu = 0$$

Variance $\sigma^2 = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx - \mu^2$

$$= \int_{-\infty}^{\infty} x^2 \cdot ce^{-|x|}\,dx - \mu^2$$

$$= c\int_{-\infty}^{\infty} x^2 \cdot e^{-|x|}\,dx - \mu^2$$

$$= 2c\int_{-\infty}^{\infty} x^2 \cdot e^{-|x|}\,dx - 0$$

$$= 2\times\frac{1}{2}\int_{0}^{\infty} x^2 \cdot e^{-|x|}\,dx$$

$$= \left[x^2\cdot\frac{e^{-x}}{-1} - 2x\cdot\frac{e^{-x}}{1} + 2\cdot\frac{e^{-x}}{-1}\right]_0^{\infty}$$

$$= \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^{\infty}$$

$$= \left[0-(0-0-2(1))\right]$$

$$\sigma^2 = 2$$

iii) To find $P(0 \le x \le 4)$:-

$$P(0 \le x \le 4) = \int_0^4 f(x)\,dx$$

$$= \int_0^4 c \cdot e^{-|x|}\,dx$$

$$= c\int_0^4 e^{-x}\,dx$$

$$= \frac{1}{2}\left[\frac{e^{-x}}{-1}\right]_0^4 = \frac{1}{2}\left[-e^{-4} - (-e^{0})\right]$$

$$= 0.4908$$

10) If x is a continuous random variable and $y = ax+b$ Prove that $E(y) = aE(x)+b$ & $V(y) = a^2 V(x)$, where V stands for variance & a, b are constants.

Sol:- proof:-

Given $y = ax+b$

we know that $E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$

Similarly $E(y) = \int_{-\infty}^{\infty} y \cdot f(x)\,dx$

Replace $y = ax+b$

$$E(ax+b) = \int_{-\infty}^{\infty} (ax+b) \cdot f(x)\,dx$$

$$= \int_{-\infty}^{\infty} ax\, f(x)\,dx + \int_{-\infty}^{\infty} b \cdot f(x)\,dx$$

$$= a\int_{-\infty}^{\infty} x\, f(x)\,dx + b\int_{-\infty}^{\infty} f(x)\,dx$$

$$= a \cdot E(x) + b(1)$$

$$\therefore E(y) = a[E(x)] + b$$

To prove $V(y) = a^2 \cdot V(x)$:-

we know that $V(x) = E(x^2) - [E(x)]^2$

$$V(y) = E(y^2) - [E(y)]^2$$

Replace $y = ax+b$

$$V(ax+b) = E[(ax+b)^2] - [E(ax+b)]^2$$

$$= E[a^2x^2 + b^2 + 2abx] - [aE(x)+b]^2$$

$$= a^2 \cdot E(x^2) + \cancel{b^2} + 2ab\,E(x) - [a^2[E(x)]^2 + \cancel{b^2} + 2ab\,E(x)]$$

$$= a^2 \cdot \left[E(x^2) - [E(x)]^2\right]$$

$$= a^2 \cdot V(x)$$

$$\therefore V(y) = a^2 V(x)$$

11) A continuous random variable whose probability density function $f(x) = \begin{cases} kxe^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$. Find i) k ii) Mean iii) variance.

Sol:-

i) we know that $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\Rightarrow \int_{-\infty}^{0} 0\, dx + \int_{0}^{\infty} kxe^{-\lambda x}\, dx = 1$$

$$0 + k\int_{0}^{\infty} x \cdot e^{-\lambda x}\, dx = 1$$

$$k\left[x \cdot \frac{e^{-\lambda x}}{-\lambda} - 1 \cdot \frac{e^{-\lambda x}}{(-\lambda)^2}\right]_0^{\infty} = 1$$

$$k\left[0 - \left(0 - \frac{1}{\lambda^2}\right)\right] = 1$$

$$\frac{k}{\lambda^2} = 1$$

$$k = \lambda^2$$

ii) Mean $\mu = \int_{-\infty}^{\infty} x f(x)\, dx$

$$= \int_{-\infty}^{0} 0\, dx + \int_{0}^{\infty} x \cdot kxe^{-\lambda x}\, dx$$

$$= k\int_{0}^{\infty} x^2 e^{-\lambda x}\, dx$$

$$= k\left[x^2 \cdot \frac{e^{-\lambda x}}{-\lambda} - 2x \cdot \frac{e^{-\lambda x}}{\lambda^2} + 2\frac{e^{-\lambda x}}{-\lambda^3}\right]_0^{\infty}$$

$$= k\left[0 - \left(0 - 0 + \frac{2}{\lambda^3}\right)\right]$$

$$\mu = \frac{2\lambda^2}{\lambda^3} = \frac{2}{\lambda}$$

iii) variance $\sigma^2 = \int_{-\infty}^{\infty} x^3 f(x)\,dx - \mu^2$

$$= 0 + k\int_0^\infty x^3 e^{-\lambda x}\,dx - \mu^2$$

$$= k\left[x^3\cdot\frac{e^{-\lambda x}}{-\lambda} - 3x^2\cdot\frac{e^{-\lambda x}}{\lambda^2} + 6x\cdot\frac{e^{-\lambda x}}{-\lambda^3} - 6\cdot\frac{e^{-\lambda x}}{\lambda^4}\right]_0^\infty$$

$$+ \cancel{24\frac{e^{-\lambda x}}{-\lambda^5}\Big]_0^\infty} - \left(\frac{2}{\lambda}\right)^2$$

$$= k\left[0 - \left(\frac{-6}{+\lambda^4}\right)\right] - \frac{4}{\lambda^2}$$

$$= \cancel{\frac{24\times 2}{\lambda^5}} - \cancel{\frac{4}{\lambda^2}}$$

$\frac{2}{\lambda^2}$

$$\cancel{\sigma^2 = \frac{48-4\lambda^4}{\lambda^6}}$$

$$\sigma^2 = k\left(+\frac{6}{\lambda^4}\right) - \frac{4}{\lambda^2}$$

$$= \lambda^2\left(\frac{+6}{\lambda^4}\right) - \frac{4}{\lambda^2}$$

$$= \frac{6-4}{\lambda^2}$$

$$\sigma^2 = 2/\lambda^2$$

Date 24/1/15

12) If X is a continuous random variable with probability density function $f(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ 0 & \text{else where} \end{cases}$ If $P(a \le X \le 1) = \frac{19}{81}$ find the value of a.

Sol:- Given $P(a \le X \le 1) = \frac{19}{81}$

$$\int_a^1 f(x)\,dx = \frac{19}{81}$$

$$\int_a^1 x^2\,dx = \frac{19}{81}$$

$$\frac{1}{3}\left[x^3\right]_a^1 = \frac{19}{81}$$

$$\frac{1}{3}(1-a^3) = \frac{19}{81}$$

$$1-a^3 = \frac{57}{81}$$

$$a^3 = 1-\frac{57}{81}$$

$$a = \left(\frac{24}{81}\right)^{1/3}$$

$$a = \frac{2}{3} = 0.66$$

13) Let X denote the random variable has the probability function

x_i	-3	-2	-1	0	1	2	3
$P(x_i)$	k	0.1	k	0.2	2k	0.4	2k

Find i) K ii) Mean iii) variance.

Sol:-

i) sum of probabilities = 1

$$k+0.1+k+0.2+2k+0.4+2k = 1$$

$$6k+0.7 = 1$$

$$6k = 1-0.7$$

$$k = \frac{0.3}{6}$$

$$k = 0.05$$

ii) Mean $\mu = \sum_{i=-3}^{3} x_i\, P(x_i)$

$$= -3(k) - 2(0.1) - 1(k) + 0 + 1(2k) + 2(0.4) + 3(2k)$$

$$= -4k + 8k - 0.2 + 0.8$$

$$= +4k + 0.6$$

$$= +4(0.05) + 0.6$$

$$= 0.8$$

iii) variance $\sigma^2 = \sum x^2 P(x) - \mu^2$

$$= 2.86$$

Binomial Distribution:- A random variable X has the binomial distribution if it assume only a non-negative values and the probability density function is given by P(X)

$$P(X=x) = P(x) = \begin{cases} n_{c_x} \cdot p^x \cdot q^{n-x} & , x = 0,1,\text{----} n \\ 0 & , \text{otherwise} \end{cases}$$

, and

the probability mass function $P(x) = \sum n_{c_x} \cdot p^x \cdot q^{n-x}$ where $q = 1-p$

Mean of Binomial distribution:-

We know that $\mu = E(x) = \sum_{x=0}^{n} x\left[n_{c_x} \cdot p^x \cdot q^{n-x}\right]$

$$= 0 + \sum_{n=1}^{n} x\left(n_{c_x} p^x \cdot q^{n-x}\right)$$

$$= (1)\left[n_{c_1} p^1 q^{n-1}\right] + 2\left[n_{c_2} p^2 q^{n-2}\right] + 3\left[n_{c_3} p^3 q^{n-3}\right] + \cdots$$

$$= n p \cdot q^{n-1} + \cancel{2} \cdot \frac{n(n-1)}{\cancel{2!}} p^2 q^{n-2} + 3 \cdot \frac{n(n-1)(n-2)}{3!} p^3 q^{n-3} + \cdots$$

$$= np\left[q^{n-1} + (n-1) p \cdot q^{n-2} + \frac{(n-1)(n-2)}{2} p^2 \cdot q^{n-3} + \cdots\right]$$

$$= np\left[(n-1)_{c_0} q^{n-1} + (n-1)_{c_1} p \cdot q^{n-2} + (n-1)_{c_2} p^2 \cdot q^{n-3} + \cdots\right]$$

$$\left\{(x+y)^n = n_{c_0} x^n + n_{c_1} x^{n-1} \cdot y + n_{c_2} (x)^{n-2} \cdot y^2 + \cdots\right\}$$

$$= np\left[q+p\right]^{n-1} \quad (\because p+q = 1)$$

$$= np(1)^{n-1}$$

$$= np$$

$$\boxed{\mu = np}$$

Variance of Binomial distribution:-

We know that $V(x) = \sigma^2 = \sum_{x=0}^{n} x^2 \, n_{c_x} \cdot p^x q^{n-x} - \mu^2$

$$= \sum_{x=0}^{n} (x^2 - x + x)\, n_{c_x} \cdot p^x q^{n-x} - \mu^2$$

$$= \sum_{x=0}^{n} \left[x(x-1) + x\right] n_{c_x} p^x \cdot q^{n-x} - \mu^2$$

$$= \sum_{x=0}^{n} \left[x(x-1)\, n_{c_x} p^x \cdot q^{n-x}\right] + \sum_{x=0}^{n} x \cdot n_{c_x} p^x q^{n-x} - \mu^2$$

$$= 0 + 0 + 2(2-1) n_{c_2} p^2 q^{n-2} + 3(3-1) n_{c_3} p^3 q^{n-3} + 4(4-1) n_{c_4} p^4 q^{n-4}$$

$$+ \cdots + \sum x P(x) - \mu^2$$

$$= \cancel{2} \cdot \frac{n(n-1)}{\cancel{2!}} p^2 q^{n-2} + \cancel{3}(\cancel{2}) \frac{n(n-1)(n-2)}{\cancel{3!}} p^3 q^{n-3} + \cancel{4}(\cancel{3}) \frac{n(n-1)(n-2)(n-3)}{\cancel{4!}\cdot 2} p^4 q^{n-4} + \cdots + \mu - \mu^2$$

$$\sigma^2 = n(n-1)p^2\left[q^{n-2} + (n-2) p \cdot q^{n-3} + \frac{(n-2)(n-3)}{2!} p^2 q^{n-4} + \cdots\right]$$

$$+ \mu - \mu^2$$

$$= n(n-1)p^2\left[(n-2)_{c_0}\, q^{n-2} + (n-2)_{c_1}\, p \cdot q^{n-3} + (n-2)_{c_2}\, p^2 q^{n-4} + \ldots\right] + \mu - \mu^2$$

$$= n(n-1)p^2\left[q+p\right]^{n-2} + \mu - \mu^2$$

$$= \cancel{n^2p^2} - np^2(1) + np - \cancel{n^2p^2}$$

$$= np - np^2$$

$$= np(1-p)$$

$$\boxed{\sigma^2 = npq} \qquad \left(\because p+q=1,\ 1-p=q\right)$$

Standard deviation of Binomial distribution:-

$$\sigma = \sqrt{V(x)}$$

$$\boxed{\sigma = \sqrt{npq}}$$

Recurrence relation for Binomial distribution:-

$$P(x+1) = \frac{(n-x)\,p}{(x+1)\,q} \times P(x)$$

Prob:-

1) Ten coins are thrown simultaneously. Find the probability of getting i) atleast 7 heads ii) atleast 6 heads.

Sol:- Given data,

Let p = probability of getting a head $= \frac{1}{2}$

q = probability of getting a tail $= \frac{1}{2}$

$n = 10$

We know that $P(X=x) = n_{c_x} \cdot p^x \cdot q^{n-x}$

i) **Atleast 7 heads:-**

$$P(X \geq 7) = ?$$

$$P(X \geq 7) = P(x=7) + P(x=8) + P(x=9) + P(x=10)$$

$$= {}^{10}c_7 \cdot \left(\tfrac{1}{2}\right)^7 \cdot \left(\tfrac{1}{2}\right)^3 + {}^{10}c_8 \left(\tfrac{1}{2}\right)^8 \left(\tfrac{1}{2}\right)^2 + {}^{10}c_9 \left(\tfrac{1}{2}\right)^9 \cdot \left(\tfrac{1}{2}\right) + {}^{10}c_{10} \left(\tfrac{1}{2}\right)^{10} \left(\tfrac{1}{2}\right)^0$$

$$= \left(\tfrac{1}{2}\right)^{10}\left[120 + 45 + 10 + 1\right]$$

$$= \frac{11}{64}$$

iii) At least 6 heads:-

$P(x \geq 6) = P(x=6) + P(x \geq 7)$

$= {}^{10}C_6 \left(\frac{1}{2}\right)^6 \cdot \left(\frac{1}{2}\right)^4 + \frac{11}{64}$

$= \frac{193}{512}$

2) Two dice are thrown 5 times. Find the probability of getting 7 as sum i) atleast once ii) 2 times iii) $P(1 < x < 5)$

Sol:- P = probability of getting a sum 7 in a 2 dice is $\frac{6}{36}$

$P = \frac{6}{36} = \frac{1}{6}$

$q = 1 - P = 1 - \frac{1}{6}$

$= \frac{5}{6}$

$n = 5$

i) Atleast once:-

$P(x \geq 1) = ?$

$P(x \geq 1) = P(x=1) + P(x=2) + P(x=3) + P(x=4) + P(x=5)$

(or)

$= 1 - P(x=0)$

$= 1 - \left[{}^5C_0 \, P\left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^5\right]$

$= 1 - \left[1(1)\left(\frac{5}{6}\right)^5\right]$

$= 0.598$

ii) 2 times:-

$P(x=2) = {}^5C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^3$

$= 10\left(\frac{1}{36}\right)\left(\frac{125}{216}\right)$

$= 0.160$

iii) $P(1 < x < 5)$:-

$P(x=2) + P(x=3) + P(x=4)$

$= {}^5C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^3 + {}^5C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2 + {}^5C_4 \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^1$

$= 0.160 + 0.032 + 0.0032$

$= 0.196$

3) Out of 800 families with 5 children each, how many would you expect to have a) 3 boys b) 5 girls c) either 2 or 3 boys d) Atleast one boy. Assume equal probabilities for boy and girl.

Sol:- Given that equal probabilities for boy and girl $= \frac{1}{2}$

i.e., p = probability of boy $= \frac{1}{2}$

q = probability of girl $= \frac{1}{2}$

no. of children in each family $n = 5$

a) 3 boys:-

$$P(x=3) = n_{c_3} (p)^3 (q)^2$$

$$= 5_{c_3} \left(\tfrac{1}{2}\right)^3 \left(\tfrac{1}{2}\right)^2$$

$$= \frac{5}{16}$$

In 800 families $= \frac{5}{16} \times 800$

$= 250$

b) 5 girls:-

$$P(5 \text{ girls}) = P(x=0) = 5_{c_0} \left(\tfrac{1}{2}\right)^0 \left(\tfrac{1}{2}\right)^5$$

$$= \frac{1}{32}$$

In 800 families $= \frac{1}{32} \times 800$

$= 25$

c) Either 2 or 3 boys:-

$$P(x=2) + P(x=3) = 5_{c_2} \left(\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2}\right)^3 + \left(5_{c_3}\right) \left(\tfrac{1}{2}\right)^3 \left(\tfrac{1}{2}\right)^2$$

$$= 10\left(\frac{1}{32}\right) + 10\left(\frac{1}{32}\right)$$

$$= \frac{5}{8}$$

In 800 families $= 500$

d) Atleast one boy :-

$$P(X \geq 1) = 1 - P(X=0)$$

$$= 1 - {}^{5}C_{0} (p)^{0} (q)^{5-0}$$

$$= 1 - 1\left(\frac{1}{2}\right)^{0}\left(\frac{1}{2}\right)^{5}$$

$$= 1 - \frac{1}{32}$$

$$= \frac{31}{32}$$

In 800 families = 775

4) The mean and variance of Binomial distribution is 4 and $\frac{4}{3}$. Find $P(X \geq 1)$

Sol:- Given mean $\mu = 4$

Variance $\sigma^2 = \frac{4}{3}$

$$np = 4, \quad npq = \frac{4}{3}$$

$$4q = \frac{4}{3}$$

$$q = \frac{1}{3}$$

$$p = \frac{2}{3}$$

$$np = 4$$

$$n\left(\frac{2}{3}\right) = 4$$

$$n = 6$$

$$P(X \geq 1) = 1 - P(X=0)$$

$$= 1 - {}^{6}C_{0}\left(\frac{2}{3}\right)^{0}\left(\frac{1}{3}\right)^{6}$$

$$= 1 - \left(\frac{1}{3}\right)^{6}$$

$$= \frac{728}{729}$$

5) The probability that a man hitting target is $\frac{1}{3}$. If he fires 6 times, find the probability that i) Atmost 5 times ii) Exactly one time iii) Atleast two times.

23

Sol:- Given $p = \frac{1}{3}$, $q = \frac{2}{3}$, $n = 6$

i) Atmost 5 times:- $P(X \leq 3) = P(X=3) + P(X=2) + P(X=1)$

$$= {}^{6}C_{3}\left(\frac{1}{3}\right)^{3}\left(\frac{2}{3}\right)^{3} + {}^{6}C_{2}\left(\frac{1}{3}\right)^{2}\left(\frac{1}{3}\right)^{4} + {}^{6}C_{1}\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^{5}$$

$= 0.998$

(or)

$1 - P(X > 5)$

$= 1 - P(X \geq 6)$

$= 1 - P(X = 6)$

$= (0.998)$

ii) **Exactly one time:-**

$$P(X=1) = 6C_1 \left(\tfrac{1}{3}\right)^1 \left(\tfrac{2}{3}\right)^5$$

$= 0.2633$

iii) **Atleast 2 times:-**

$$P(X \geq 2) = 1 - [P(X=0) + P(X=1)]$$

$$= 1 - \left[6C_0 \left(\tfrac{1}{3}\right)^0 \left(\tfrac{2}{3}\right)^6 + 6C_1 \left(\tfrac{1}{3}\right)^1 \left(\tfrac{2}{3}\right)^5\right]$$

$= 0.6488$

Date 28/1/15

6) 20% of items produced from a factory are defective. Find the probability that in a sample of 5 choosen at random i) None is defective ii) one is defective iii) Find $P(1<x<5)$

Sol:- probability of defective items $p = 20\% = 0.2$

$q = 1 - p = 1 - 0.2$
$= 0.8$

$n = 5$

i) $P(X=0) = ?$

$$P(X=0) = 5C_0 (0.2)^0 (0.8)^5$$

$= 0.3276$

ii) $$P(X=1) = 5C_1 (0.2)^1 (0.8)^4$$

$= 0.4096$

iii) $$P(1<X<5) = 5C_2 (0.2)^2 (0.8)^3 + 5C_3 (0.2)^3 (0.8)^2 + 5C_4 (0.2)^4 (0.8)$$

$= 0.2048 + 0.0512 + 0.0064$

$= 0.2624$

*** Dec

Poisson's Distribution (P.D):-

A random variable x is said to be follow a poisson distribution if it assume non-negative values and the probability density function is given by

$$P(X=x) = P(x) = \begin{cases} \dfrac{e^{-\lambda}\cdot\lambda^{x}}{x!} , & x = 0,1,2 \cdots \\ 0 , & \text{otherwise} \end{cases}$$

here λ is called the parameter of poisson distribution.

Mean of the poisson Distribution:-

we know that $\mu = E(x) = \Sigma x \cdot p(x)$

$$= \sum_{x=0}^{\infty} (x) \frac{e^{-\lambda}\cdot\lambda^{x}}{x!}$$

$$= 0 + \sum_{x=1}^{\infty} \frac{x e^{-\lambda}\cdot\lambda^{x}}{x(x-1)!}$$

put $x-1 = y$

limits:- $x = 1 \Rightarrow y = 0$

$x = \infty \Rightarrow y = \infty$

$$\mu = \sum_{y=0}^{\infty} \frac{e^{-\lambda}\cdot(\lambda)^{y+1}}{y!}$$

$$= \lambda \sum_{y=0}^{\infty} \frac{e^{-\lambda}\cdot\lambda^{y}}{y!}$$

$$= \lambda e^{-\lambda} \sum_{y=0}^{\infty} \left[\frac{\lambda^{y}}{y!}\right]$$

$$= \lambda e^{-\lambda}\left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right]$$

$$\mu = \lambda e^{-\lambda}(e^{\lambda}) \qquad \left(\because e^{x} = 1 + x + \frac{x^2}{2} + \cdots\right)$$

$$\boxed{\mu = \lambda}$$

Variance of poisson Distribution:-

we know that $\sigma^2 = V(x) = \Sigma x^2 p(x) - \left[\Sigma x \cdot p(x)\right]^2$

$$= \sum_{x=0}^{\infty} \frac{x^2 \cdot e^{-\lambda}\lambda^{x}}{x!} - \mu^2$$

$$= 0 + \sum_{x=1}^{\infty} \frac{x^2\cdot e^{-\lambda}\lambda^{x}}{x(x-1)!} - \mu^2$$

$$= \sum_{x=1}^{\infty} \frac{x e^{-\lambda}\cdot\lambda^{x}}{(x-1)!} - \mu^2$$

$$\sigma^2 = \sum_{x=1}^{\infty} \frac{(x-1+1)\, e^{-\lambda} \lambda^x}{(x-1)!} - \mu^2$$

$$= \left\{\sum_{x=1}^{\infty} \frac{(x-1)\, e^{-\lambda}\cdot \lambda^x}{(x-1)!}\right\} + \left\{\sum_{x=1}^{\infty} \frac{e^{-\lambda}\cdot\lambda^x}{(x-1)!} - \mu^2\right\}$$

$\Rightarrow$ put $x = 1$

$$\Rightarrow 0 + \sum_{x=2}^{\infty} \frac{(x-1)\, e^{-\lambda}\cdot \lambda^x}{(x-1)!} + \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-1)!} - \mu^2$$

$$= \sum_{x=2}^{\infty} \frac{e^{-\lambda}\cdot\lambda^x}{(x-2)!} + \sum_{x=1}^{\infty} \frac{e^{-\lambda}\cdot\lambda^x}{(x-1)!} - \mu^2$$

put $x-2 = z$ & $x-1 = y$.

Limits :- $x = 2 \Rightarrow z = 0$; $x = 1 \Rightarrow y = 0$

$x = \infty \Rightarrow z = \infty$; $x = \infty \Rightarrow y = \infty$

$$\sum_{z=0}^{\infty} \frac{e^{-\lambda}\cdot\lambda^{z+2}}{z!} + \sum_{y=0}^{\infty} \frac{e^{-\lambda}\cdot\lambda^{y+1}}{y!} - \mu^2$$

$$= e^{-\lambda}\cdot\lambda^2\left[\sum_{z=0}^{\infty}\frac{\lambda^z}{z!}\right] + e^{-\lambda}\cdot\lambda\left[\sum_{y=0}^{\infty}\frac{\lambda^y}{y!}\right] - \mu^2$$

$$= e^{-\lambda}\cdot\lambda^2\left[1+\frac{\lambda}{1!}+\frac{\lambda^2}{2!}+\cdots\right] + e^{-\lambda}\cdot\lambda\left[1+\lambda+\frac{\lambda^2}{2!}+\cdots\right] - \mu^2$$

$$= e^{-\lambda}\cdot\lambda^2\cdot e^{\lambda} + e^{-\lambda}\cdot\lambda\cdot e^{\lambda} - \lambda^2$$

$$\boxed{\sigma^2 = \lambda}$$

Standard deviation of the poisson distribution is

$$\boxed{\sigma = \sqrt{\lambda}}$$

*** **Recurrence relation of the poisson Distribution :-**

$$P(x+1) = \frac{\lambda}{(x+1)}\cdot P(x)$$

Prob :-

1) A hospital switch board received an average (P.D) of 4 emergency calls in 10 mins interval. what is the probability that i) there are atmost 2 emergency calls in 10 mins. interval ii) there are exactly 3 emergency calls in a 10 min. interval.

Sol :- Given $\mu = \lambda = 4$,

i) $P(X \leq 2) = ?$

$$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= \frac{e^{-\lambda}\cdot\lambda^0}{0!} + \frac{e^{-\lambda}\cdot\lambda^1}{1!} + \frac{e^{-\lambda}\cdot\lambda^2}{2!}$$

$$= e^{-\lambda}\left[1 + \lambda + \frac{\lambda^2}{2!}\right]$$

$$= e^{-4}\left[1 + 4 + \frac{16}{2}\right]$$

$$= 0.2381$$

ii) $P(X=3) \Rightarrow ?$

$$P(X=3) = \frac{e^{-\lambda}\cdot\lambda^3}{3!}$$

$$= \frac{e^{-4}\cdot 4^3}{3!}$$

$$= 0.1953$$

Date 29/1/15

2) It has been found that 2% of the tools produced by certain machine are defective. What is the probability that in a shipment of 400 such tools i) 3% or more ii) 2% or less will be defective

Sol:- Given $n = 400$

$p = 2\% = 0.02$

We know that $\mu = np = \lambda$

$\Rightarrow 400(0.02) = \lambda$

$\Rightarrow \lambda = 8$

i) 3% or more :-

$\left(\because \; 100 \to 3, \; 400 \to 12\right)$

$$P(X \geq 3\%) = P(X \geq 12) = ?$$

$$= 1 - P(X < 12)$$

$$= 1 - \left[P(X=0) + P(X=1) + \cdots + P(X=11)\right]$$

$$= 1 - \left[\frac{e^{-\lambda}\cdot\lambda^0}{0!} + \frac{e^{-\lambda}\cdot\lambda^1}{1!} + \frac{e^{-\lambda}\cdot\lambda^2}{2!} + \cdots + \frac{e^{-\lambda}\cdot\lambda^{11}}{11!}\right]$$

$$= 1 - e^{-\lambda}\left[1 + \frac{\lambda}{1} + \frac{\lambda^2}{2} + \frac{\lambda^3}{6} + \cdots + \frac{\lambda^{11}}{11!}\right]$$

$$= 0.1119$$

ii) 2% or less :-

$$P(x \le 2\%) = P(x \le 8)$$

$$= P(x=0)+P(x=1)+ \cdots + P(x=8)$$

$$= \frac{e^{-\lambda}.\lambda^0}{0!} + \frac{e^{-\lambda}.\lambda^1}{1!} + \cdots + \frac{e^{-\lambda}.\lambda^8}{8!}$$

$$= e^{-\lambda}\left[1+\lambda+\frac{\lambda^2}{2!}+\cdots+\frac{\lambda^3}{8!}\right]$$

$$= 0.5925$$

3) Using Recurrence relation for the poisson distribution, find the probabilities when $x = 0,1,2,3,4,5$ if the mean of the poisson distribution is 3.

Sol:- Given mean $(\lambda) = 3$

By the definition of Recurrence formula, we have

$$P(x+1) = \frac{\lambda}{x+1}.P(x)$$

put $x=0 \Rightarrow$ $P(0+1) = \frac{\lambda}{0+1}.P(0)$

$$\Rightarrow P(1) = \frac{3}{1}\left[\frac{e^{-\lambda}.\lambda^0}{0!}\right]$$

$$P(1) = 3e^{-3}$$

put $x=1 \Rightarrow$ $P(1+1) = \frac{\lambda}{1+1}.P(1)$

$$\Rightarrow P(2) = \frac{3}{2}\left[3.e^{-3}\right]$$

$$= \frac{9}{2}e^{-3}$$

put $x=2 \Rightarrow$ $P(2+1) = \frac{\lambda}{2+1}.P(2)$

$$P(3) = \cancel{\frac{3}{3}}\left(\frac{9}{2}e^{-3}\right)$$

$$= \frac{9}{2}e^{-3}$$

put $x=3 \Rightarrow$ $P(3+1) = \frac{\lambda}{3+1}.P(3)$

$$P(4) = \frac{3}{4}.\frac{9}{2}e^{-3}$$

$$= \frac{27}{8}e^{-3}$$

put $x=4 \Rightarrow$ $P(4+1) = \frac{\lambda}{4+1}.P(4)$

$$= \frac{3}{5}.\frac{27}{8}e^{-3}$$

$$P(5) = \frac{81}{40} e^{-3}$$

put $x = 5 \Rightarrow P(5+1) = \frac{1}{5+1} \cdot P(5)$

$$P(6) = \frac{3}{6} \cdot \frac{81}{40} e^{-3}$$

$$= \frac{81}{80} e^{-3}$$

4) If a random variable x has a poisson distribution such that $P(1) = P(2)$. find i) mean ii) $P(4)$ iii) $P(x \geq 1)$ iv) $P(1 \leq x < 4)$

Sol:- Given $P(1) = P(2)$

$$\frac{e^{-\lambda} \cdot \lambda}{1!} = \frac{e^{-\lambda} \cdot \lambda^2}{2!}$$

$$\lambda = 2$$

i)

ii) $$P(4) = \frac{e^{-\lambda} \cdot \lambda^4}{4!}$$

$$= \frac{e^{-2} \cdot (2)^4}{4!} = 0.09$$

iii) $$P(x \geq 1) = 1 - P(x < 1)$$

$$= 1 - P(x = 0)$$

$$= 1 - \frac{e^{-\lambda} \cdot \lambda^0}{0!}$$

$$= 1 - e^{-2}$$

$$= 0.864$$

iv) $$P(1 \leq x < 4) = P(x=1) + P(x=2) + P(x=3)$$

$$= \frac{e^{-\lambda} \cdot \lambda}{1!} + \frac{e^{-\lambda} \cdot \lambda^2}{2!} + \frac{e^{-\lambda} \cdot \lambda^3}{3!}$$

$$= 0.721$$

5) If a poisson distribution $P(x=1)\frac{3}{2} = P(x=3)$. find i) $P(x \geq 1)$ ii) $P(x \leq 3)$ iii) $P(2 \leq x \leq 5)$

Sol:- Given $P(x=1)\frac{3}{2} = P(x=3)$

$$3P(1) = 2P(3)$$

$$3 \frac{e^{-\lambda} \cdot \lambda}{1!} = 2 \cdot \frac{e^{-\lambda} \cdot \lambda^{3}}{3!}$$