UNIT V:

Fundamental Principles of Counting:
The rules of Sum and Product, Permutations, Combinations: The Binomial Theorem, Combinations with Repetition

The Principle of Inclusion and Exclusion:
The Principle of Inclusion and Exclusion, Generalizations of Principle, Derangements: Nothing is in Its Right Place, Rook Polynomials, Arrangements with Forbidden Positions

Generating Functions:
Introductory Examples, Definitions and Examples: Calculation Techniques, Partitions of Integers, The Exponential Generating Functions, The Summation Operator.

5.1 Fundamental Principles of Counting:

5.1.1 The rules of Sum and Product

If X is a set, let us use |X| to denote the number of elements in X.

Two Basic Counting Principles:
Two elementary principles act as "building blocks" for all counting problems.

The first principle says that the whole is the sum of its parts.

Sum Rule:

If a set X is the union of disjoint nonempty subsets S1, …..,Sn,

then | X | = | S1 | + | S2 | + ….. + | Sn |.

We emphasize that the subsets S1, S2, ….,Sn must have no elements in common. Moreover, since X = S1 U S2 U ……U Sn, each element of X is in exactly one of the subsets Si. In other words, S1, S2, ….,Sn is a partition of X.

If the subsets S1, S2, ….,Sn were allowed to overlap, then a more profound principle will be needed the principle of inclusion and exclusion.

The difference is largely in semantics, for if A is an event, we can let X be the set of ways that A can happen and count the number of elements in X. Nevertheless, let us state the sum rule for counting events.

If E1, ……, En are mutually exclusive events, and E1 can happen e1 ways, E2 happen e2 ways,…. ,En can happen en ways, E1 or E2 or …. or En can happen e1 + e2 + …….. + en ways.

Again we emphasize that mutually exclusive events E1 and E2 mean that E1 or E2 can happen but both cannot happen simultaneously.

The sum rule can also be formulated in terms of choices: If an object can be selected from a reservoir in e1 ways and an object can be selected from a separate reservoir in e2 ways and an object can be selected from a separate reservoir in e2 ways, then the selection of one object from either one reservoir or the other can be made in e1 + e2 ways.

Product Rule:

If S1, …..,Sn are nonempty sets, then the number of elements in the Cartesian product is S1 x S2 x ….. xSn is the product ∏in=1 |S i |.

That is, |1 S1 x S2 x . . . . x Sn| = ∏in=1| S i |.

Observe that there are 5 branches in the first stage corresponding to the 5 elements of S1 and to each of these branches there are 3 branches in the second stage corresponding to the 3 elements of S2 giving a total of 15 branches altogether. Moreover, the Cartesian product S1 x S2 can be partitioned as (a1 x S2) U (a2 x S2) U (a3 x S2) U (a4 x S2) U (a5 x S2), where (ai x S2)

= {(ai, b1), (ai, b2), (ai, b3)}.

Thus, for example, (a3 x S2) corresponds to the third branch in the first stage followed by each of the 3 branches in the second stage.

More generally, if a1,….., an are the n distinct elements of S1 and b1,….,bm are the m distinct elements of S2, then S1 x S2 = Uin =1 (ai x S2).For if x is an arbitrary element of S1 x S2 , then x = (a, b) where a Î S1 and b Î S2.

Thus, a = ai for some i and b = bj for some j. Thus, x = (ai, bj) Î(ai x S2) and therefore x Î Uni =1(ai x S2).

Conversely, if x Î Uin =1(ai x S2), then x Î (ai x S2) for some i and thus x = (ai, bj) where bj is some element of S2. Therefore, x Î S1 x S2.

Therefore, The product rule is for two sets. The general rule follows by mathematical induction.

We can reformulate the product rule in terms of events. If events E1, E2 , …., En can Happened in e1, e2,…., and en ways, respectively, then the sequence of events E1 first, followed by E2,…., followed by En can happen e1e2 …en ways.

In terms of choices, the product rule is stated thus: If a first object can be chosen e1 ways, **a second e2 ways , …, and an nth object can be made in e1e2….en ways.**

5.1.2 Permutations & Combinations: Definition.

A permutation of n objects taken r at a time (also called an r-permutation of n objects) is an **ordered selection** or **arrangement** of r of the objects.

A combination of n objects taken r at a time (called an r-combination of n objects) is an

unordered selection of r of the objects.

Note that we are simply defining the terms r-combinations and r-permutations here and have not mentioned anything about the properties of the n objects.

For example, these definitions say nothing about whether or not a given element may appear more than once in the list of n objects.

In other words, it may be that the n objects do not constitute a set in the normal usage of the word.

SOLVED PROBLEMS

Example1. Suppose that the 5 objects from which selections are to be made are: a, a, a, b, c. then the 3-combinations of these 5 objects are :aaa, aab, aac, abc. The permutations are:

aaa, aab, aba, baa, aac, aca, caa, abc, acb, bac, bca, cab, cba.

Neither do these definitions say anything about any rules governing the selection of the r-objects: on one extreme, objects could be chosen where all repetition is forbidden, or on the other extreme, each object may be chosen up to t times, or then again may be some rule of selection between these extremes; for instance, the rule that would allow a given object to be repeated up to a certain specified number of times.

We will use expressions like {3 . a , 2. b ,5.c} to indicate either

(1) that we have 3 **+ 2 + 5 =10 objects including 3a's , 2b's and 5c's, or**

(2) that we have 3 objects a, b, c, where selections are constrained by the conditions that a can be selected at most three times, b can be selected at most twice, and c can be chosen up to five times.

The numbers 3, 2 and 5 in this example will be called repetition numbers.

Example2 The 3-combinations of {3. a, 2. b, 5.c} are: aaa, aab,

aac, abb,abc,
ccc, ccb, cca, cbb.

Example3.The 3-combinations of {3 . a, 2. b, 2. c , 1.d}are: aaa, aab,

aac, aad, bba, bbc,bbd,
cca, ccb, ccd, abc, abd, acd, bcd.

In order to include the case where there is no limit on the number of times an object can be repeated in a selection (except that imposed by the size of the selection) we use the symbol ∞ as a repetition number to mean that an object can occur an infinite number of times.

Example 4. The 3-**combinations of {∞. a, 2.b, ∞.c} are the same as in Example 2 even though** a and c can be repeated an infinite number of times. This is because, in 3-combinations, 3 is the limit on the number of objects to be chosen.

If we are considerin**g selections where each object has ∞ as its repetition number then** we designate such selections as selections with unlimited repetitions. In particular, as election of r objects in this case will be called r-combinations with unlimited repetitions and any ordered arrangement of these r objects will be an r-permutation with unlimited repetitions.

Example5 The combinations of a ,b, c, d with unlimited repetitions are the 3-combinations **of {∞ . a , ∞. b, ∞. c, ∞. d}. These are 20 such 3**-combinations, namely: aaa,

aab, aac, aad,

bbb, bba, bbc, bbd,
ccc, cca, ccb, ccd,
ddd, dda, ddb, ddc,
abc, abd, acd, bcd.

Moreover, there are 43 = 64 of 3-permutations with unlimited repetitions since the first position can be filled 4 ways (with a, b, c, or d), the second position can be filled 4 ways, and likewise for the third position.

The 2-**permutations of {∞. a, ∞. b, ∞. c, ∞. d} do not present such a formi**dable list and so we tabulate them in the following table.

2-combinations with Unlimited Repetitions	2-permutations with Unlimited Repetitions
Aa	aa
Ab	ab, ba
Ac	ac, ca
Ad	ad, da
Bb	bb
Bc	bc, cb
Bd	bd, db
Cc	cc
Cd	cd, dc
Dd	dd
10	16

Of course these are not the only constraints that can be placed on selections

The possibilities are endless. We list some more examples just for concreteness. We might, for example, consider selections o**f {∞.a, ∞. b, ∞. c} where b can be chosen only even** number of times.

Thus, 5-combinations with these repetition numbers & this constraint would be those 5-Combinations with unlimited repetitions and where b is chosen 0, 2, or 4 times.

Example6 The 3-**combinations of {∞ .a, ∞ .b, 1 .c,1 .d} where b can be chosen only an even** number of times are the 3-combinations of a, b, c, d where a can be chosen up 3 times, b can be chosen 0 or 2 times, and c and d can be chosen at most once. The 3-cimbinations subject to these constraints are:

aaa, aac, aad, bbc, bbd, acd.

As another example, we might be interested in, selections of {∞.a, 3.b, 1.c} where a can be chosen a prime number of times. Thus, the 8-combinations subject to these constraints would be all those 8-combinations where a can be chosen 2, 3, 5, or 7 times, b can chosen up to 3 times,

and c can be chosen at most once.

There are, as we have said, an infinite variety of constraints one could place on selections. You can just let your imagination go free in conjuring up different constraints on the selection, would constitute an r-combination according to our definition. Moreover, any arrangement of these r objects would constitute an r-permutation.

While there may be an infinite variety of constraints, we are primarily interested in two major types: one we have already described—combinations and permutations with unlimited repetitions, the other we now describe.

If the repetition numbers are all 1, then selections of r objects are called r-combinations without repetitions and arrangements of the r objects are r-permutations without repetitions. We remind you that r-combinations without repetitions are just subsets of the n elements containing exactly r elements. Moreover, we shall often drop the repetition number 1 when considering r-combinations without repetitions. For example, when considering r-combinations of {a, b, c, d} we will mean that each repetition number is 1 unless otherwise designated, and, of course, we mean that in a given selection an element need not be chosen at all, but, if it is chosen, then in this selection this element cannot be chosen again.

Example7. Suppose selections are to be made from the four objects a, b, c,d.

2-combinations withoutRepetitions	2-Permutations withoutRepetitions
ab	ab,ba
ac	ac, ca

ad	ad, da
bc	bc, cb
bd	bd, db
cd	cd, dc
6	12

There are six 2-combinations without repetitions and to each there are two 2-permutations giving a total of twelve 2-permutations without repetitions.

Note that total number of 2-combinations with unlimited repetitions in Example 5 included six 2-combinations without repetitions of Example.7 and as well 4 other 2-combinations where repetitions actually occur. Likewise, the sixteen 2-permutations with unlimited repetitions included the twelve 2-permutations without repetitions.

3-combinations without Repetitions abc abd acd bcd are 4

3-Permutations without Repetitions

abc, acb, bac, bca, cab, cbaabd, adb, bad, bda, dab, dbaacd, adc, cad, cda, dac, dcabcd, bdc, cbd, cdb, dbc, dcb are 24

Note that to each of the 3-combinations without repetitions there are 6 possible 3- permutations without repetitions. Momentarily, we will show that this observation can be generalized.

5.1.4 Combinations and Permutations with Repetitions:

General formulas for enumerating combinations and permutations will now be presented. At this time, we will only list formulas for combinations and permutations without repetitions or with unlimited repetitions. We will wait until later to use generating functions to give general techniques for enumerating combinations where other rules govern the selections.

Let P (n, r) denote the number of r-permutations of n elements without repetitions.

Theorem 5.3.1.(Enumerating r-permutations without repetitions).

$P(n, r) = n(n-1)\ldots\ldots(n - r + 1) = n! / (n-r)!$

Proof. Since there are n distinct objects, the first position of an r-permutation may be filled in n ways. This done, the second position can be filled in n-1 ways since no repetitions are allowed and there are n – 1 objects left to choose from. The third can be filled in n-2 ways. By applying the product rule, we conduct that

$P (n, r) = n(n-1)(n-2)\ldots\ldots(n - r + 1)$. From the definition of factorials, it follows that

$$P (n, r) = n! / (n-r)!$$

When r = n, this formula becomes

$$P (n, n) = n! / 0! = n!$$

When we explicit reference to r is not made, we assume that all the objects are to be arranged; thus we talk about the permutations of n objects we mean the case r=n.

Corollary1. There are n! permutations of n distinct objects.

Example1. There are 3! = 6 permutations of {a, b,c}.

There are 4! = 24 permutations of (a, b, c, d). The number of 2-permutations {a, b, c, d, e} is P(5, 2) = 5! / (5 - 2)! = 5 x 4 = 20. The number of 5-letter words using the letters a, b, c, d, and e at most once is P (5, 5) = 120.

Example 2 There are P (10, 4) = 5,040 4-digit numbers that contain no repeated digits since **each such number is just an arrangement of four of the digits 0, 1, 2, 3 , …., 9 (leading zeroes are** allowed). There are P (26, 3) P(10, 4) license plates formed by 3 distinct letters followed by 4 distinct digits.

Example3. In how many ways can 7 women and 3 men be arranged in a row if the 3 men must always stand next to each other?

Solution :

There are 3! ways of arranging the 3 men. Since the 3 men always stand next to each other, we treat them as a single entity, which we denote by X.

Then if W1, W2, ….., W7 represents the women, we next are interested in the number of ways of arranging {X, W1, W2, W3,……., W7}. There are 8! Permutations these 8 objects. Hence there are (3!) (8!) permutations altogether.

Example4. In how many ways can the letters of the English alphabet be arranged so that there are exactly 5 letters between the letters a and b?

There are P (24, 5) ways to arrange the 5 letters between a and b, 2 ways to place a and b, and then 20! ways to arrange any 7-letter word treated as one unit along with the remaining 19 letters. The total is P (24, 5) (20!) (2).

Permutations for the objects are being arranged in a line. If instead of arranging objects in a line, we arrange them in a circle, then the number of permutations decreases.

Example5. In how many ways can 5 children arrange themselves in a ring?

Solution: Here, the 5 children are not assigned to particular places but are only arranged relative to one another. Thus, the arrangements (see Figure 2-3) are considered the same if the children are in the same order clockwise. Hence, the position of child C1 is immaterial and it is only the position of the 4 other children relative to C1 that counts. Therefore, keeping C1 fixed in position, there are 4! arrangements of the remaining children.

Binomial Coefficients:

In mathematics, the **binomial coefficient** $\binom{n}{k}$ is the coefficient of the x^k term in the polynomial expansion of the binomial power $(1 + x)^n$. In combinatorics, $\binom{n}{k}$ is interpreted as the number of *k*-element subsets (the *k*-combinations) of an *n*-element set, that is the number of ways that *k* things can be "chosen" from a set of *n* things.

Hence, $\binom{n}{k}$ is often read as "*n* choose *k*" and is called the **choose function** of *n* and *k*. The notation was introduced by Andreas von Ettingshausenin 182, although the numbers were already known centuries before that (see Pascal's triangle). Alternative notations include C(*n*, *k*), $_nC_k$, ^nC_k, C^n_k, C^k_n, in all of which the C stands for combinations or *choices*.

For natural numbers (takentoinclude0) *n* and *k*, the binomial coefficient $\binom{n}{k}$ can be defined as the coefficient of the monomial X^k in the expansion of $(1 + X)^n$. The same coefficient also occurs (if $k \le n$) in the binomial formula

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

(valid for any elements *x*, *y* of a commutative ring), which explains the name "binomial coefficient".

Another occurrence of this number is in combinatorics, where it gives the number of ways, disregarding order, that a *k* objects can be chosen from among *n* objects; more formally, the number of *k*-element subsets (or *k*-combinations) of an *n*-element set. This number can be seen to be equal to the one of the first definition, independently of any of the formulas below to compute it: if in each of the *n* factors of the power $(1 + X)^n$ one temporarily labels the term *X* with an index *i* (running from 1 to *n*), then each subset of *k* indices gives after expansion a contribution *k* , and the coefficient of that monomial in the result will be the number of such subsets. This shows in particular that $\binom{n}{k}$ is a natural number for any natural numbers *n* and *k*. There are many other combinatorial interpretations of binomial coefficients (counting problems for which the answer is given by a binomial coefficient expression), for instance the number of words formed of *n* bits (digits 0 or 1) whose sum is *k*, but most of these are easily seen to be equivalent to counting *k*-combinations. Several methods exist to compute the value of $\binom{n}{k}$ without actually expanding a binomial power or counting *k*-combinations.

5.3.1.2 Binomial & Multinomial theorems:

Binomial theorem:

In elementary algebra, the **binomial theorem** describes the algebraic expansion of powers of a binomial. According to the theorem, it is possible to expand the power $(x + y)^n$ into a sum involving terms of the form $ax^b y^c$, where the coefficient of each term is a positive integer, and y the sum of the exponents of *x* and *y* in each term is *n*. For example,

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

The coefficients appearing in the binomial expansion are known as binomial coefficients. They are the same as the entries of Pascal's triangle, and can be determined by a simple formula involving factorials. These numbers also arise in combinatorics, where the coefficient of x,y is equal to the number of different combinations of *k* elements that can be chosen from an *n*- element set.

According to the theorem, it is possible to expand any power of *x* + *y* into a sum of the form as

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots$$
$$\cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n,$$

where $\binom{n}{k}$ denotes the corresponding binomial coefficient. Using summation notation, the formula above can be written

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

This formula is sometimes referred to as the **Binomial Formula** or the **Binomial Identity**.

A variant of the binomial formula is obtained by substituting 1 for *x* and *x* for *y*, so that it involves only a single variable. In this form, the formula reads

$$(1+x)^n = \binom{n}{0}x^0 + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \cdots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n,$$

or equivalently

$$(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k.$$

Multinomial theorem:

In mathematics, the **multinomial theorem** says how to write a power of a sum in terms of powers of the terms in that sum. It is the generalization of the binomial theorem to polynomials. For any positive integer *m* and any nonnegative integer *n*, the multinomial formula tells us how a polynomial expands when raised to an arbitrary power:

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1, k_2, \ldots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}.$$

The summation is taken over all sequences of nonnegative integer indices k_1 through k_m such the sum of all k_i is *n*. That is, for each term in the expansion, the exponents must add up to *n*.

Also, as with the binomial theorem, quantities of the form x^0 that appear are taken to equal 1 (even when *x* equals zero). Alternatively, this can be written concisely using multi indices as

$$(x_1 + \cdots + x_m)^n = \sum_{|\alpha|=n} \binom{n}{\alpha} x^\alpha$$

where α = **(**α_1**,**α_2**,…,**α_m) and x elements.

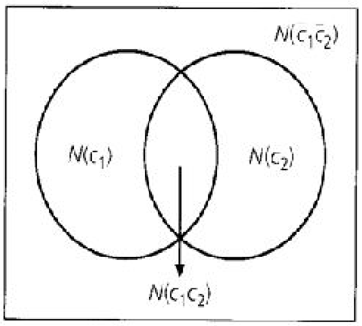
5.2.1 The principle of Inclusion and Exclusion

Let us Consider C_1 , C_2 are the 2 conditions on two sets can be represented by

$$N(\bar{c}_1\,\bar{c}_2) = N - [N(c_1) + N(c_2) + N(c_1c_2)]$$

The above equation can be represented by Venn diagram as shown below, where

$$N(\overline{c_1c_2}) \neq N(\bar{c}_1\bar{c}_2)$$



For 3 sets and 3 Conditions can be represented By

$$N(\bar{c}_1\,\bar{c}_2\,\bar{c}_3) = N - [N(c_1) + N(c_2) + N(c_3)]$$
$$+ [N(c_1c_2) + N(c_1c_3) + N(c_2c_3)] - N(c_1c_2c_3)$$

Four sets it can be represented by

$$N(\bar{c}_1\,\bar{c}_2\,\bar{c}_3\,\bar{c}_4) = N - [N(c_1) + N(c_2) + N(c_3) + N(c_4)]$$
$$+ [N(c_1c_2) + N(c_1c_3) + N(c_1c_4) + N(c_2c_3) + N(c_2c_4) + N(c_3c_4)]$$
$$- [N(c_1c_2c_3) + N(c_1c_2c_4) + N(c_1c_3c_4) + N(c_2c_3c_4)] + N(c_1c_2c_3c_4)$$

For each element x, we have five cases:

(0) x satisfies none of the four conditions;

(1) x satisfies only one of the four conditions;

(2) x satisfies exactly two of the four conditions;

(3) x satisfies exactly three of the four conditions;

(4) x satisfies all the four conditions.

1. Say x satisfies no condition. x is counted once on the left side and once on the right side.
2. Say x satisfies c_1. It is not counted on the left side. It is counted once in N and once in $N(c_1)$.
3. Say x satisfies c_2 and c_4. It is not counted on the left side. It is counted once in N, $N(c_2)$, $N(c_4)$ and $N(c_2c_4)$.
4. Say x satisfies c_1, c_2 and c_4. It is not counted on the left side. It is counted once in N, $N(c_1)$, $N(c_2)$, $N(c_4)$, $N(c_1c_2)$, $N(c_1c_4)$, $N(c_2c_4)$ and $N(c_1c_2c_4)$.

5. Say x satisfies all conditions. It is not counted on the left side. It is counted once in all the subsets on the right side.

$N(c_1\overline{c_2}\,\overline{c_3}\,\overline{c_4})$ =

$$N(c_1\overline{c}_2\overline{c}_3\overline{c}_4) = N(\overline{c}_2\overline{c}_3\overline{c}_4) - N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4)$$

$$= \{N - [N(c_2) + N(c_3) + N(c_4)] + [N(c_2c_3) + N(c_2c_4) + N(c_3c_4)] - N(c_2c_3c_4)\} - \{N - [N(c_1) + N(c_2) + N(c_3) + N(c_4)] + [N(c_1c_2) + N(c_1c_3) + N(c_1c_4) + N(c_2c_3) + N(c_2c_4) + N(c_3c_4)] - [N(c_1c_2c_3) + N(c_1c_2c_4) + N(c_1c_3c_4) + N(c_2c_3c_4)] + N(c_1c_2c_3c_4)\}, \text{ or}$$

$$N(c_1\overline{c}_2\overline{c}_3\overline{c}_4) = N(c_1) - [N(c_1c_2) + N(c_1c_3) + N(c_1c_4)] + [N(c_1c_2c_3) + N(c_1c_2c_4) + N(c_1c_3c_4)] - N(c_1c_2c_3c_4).$$

Example:

Determine the number of positive integer's n where n≤100 and n is not divisible by 2, 3 or 5.

1. Condition c_1 if n is divisible by 2.
2. Condition c_2 if n is divisible by 3.
3. Condition c_3 if n is divisible by 5.
4. Then the answer to this problem is

$$N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}) = N - [N(c_1) + N(c_2) + N(c_3)] + [N(c_1c_2) + N(c_1c_3) + N(c_2c_3)] - N(c_1c_2c_3)$$

⇨ $N(\overline{c_1c_2c_3})$.= 100-[50+33+20] =[16+10+6]-3 =26

Problem 2:

Determine the number of nonnegative integer solutions to the equation $x_1+x_2+x_3+x_4=18$ and $x_i \leq 7$ for all i. We say that a solution x_1, x_2, x_3, x_4 satisfies condition c_i if $x_i>7$.
Then the answer to this problem is

$$N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4) = S_0 - S_1 + S_2 - S_3 + S_4 = \binom{21}{18} - \binom{4}{1}\binom{13}{10} + \binom{4}{2}\binom{5}{2} - 0 + 0 = 246.$$

Theorem:

$$\overline{N} = N - \sum_{1\leq i\leq t} N(c_i) + \sum_{1\leq i<j\leq t} N(c_ic_j) - \sum_{1\leq i<j<k\leq t} N(c_ic_jc_k) + \cdots + (-1)^t N(c_1c_2c_3\cdots c_t).$$

The other possibility is that x satisfies *exactly* r of the conditions where $1 \le r \le t$. In this case x contributes nothing to $\overline{N}$. But on the right-hand side of Eq. (2), x is counted

(1) One time in N.

(2) r times in $\sum_{1 \le i \le t} N(c_i)$. (Once for each of the r conditions.)

(3) $\binom{r}{2}$ times in $\sum_{1 \le i < j \le t} N(c_i c_j)$. (Once for each pair of conditions selected from the r conditions it satisfies.)

(4) $\binom{r}{3}$ times in $\sum_{1 \le i < j < k \le t} N(c_i c_j c_k)$. (Why?)

. . . .

$(r+1)$ $\binom{r}{r} = 1$ time in $\sum N(c_{i_1} c_{i_2} \cdots c_{i_r})$, where the summation is taken over all selections of size r from the t conditions.

Consequently, on the right-hand side of Eq. (2), x is counted

$$1 - r + \binom{r}{2} - \binom{r}{3} + \cdots + (-1)^r \binom{r}{r} = [1 + (-1)]^r = 0^r = 0 \text{ times},$$

The Above Equation can be represented by using S_K

$$S_0 = N,$$
$$S_1 = [N(c_1) + N(c_2) + \cdots + N(c_t)],$$
$$S_2 = [N(c_1c_2) + N(c_1c_3) + \cdots + N(c_1c_t) + N(c_2c_3) + \cdots + N(c_{t-1}c_t)],$$

and, in general,

$$S_k = \sum N(c_{i_1} c_{i_2} \cdots c_{i_k}), \; 1 \le k \le t,$$

where the summation is taken over all selections of size k from the collection of t conditions. Hence S_k has $\binom{t}{k}$ summands in it.

Using this notation we can rewrite the result in Eq. (2) as

$$\overline{N} = S_0 - S_1 + S_2 - S_3 + \cdots + (-1)^t S_t.$$

5.2 Generalizations of the principle

Consider a set S with |s| = N and conditions satisfy some of the elements of S
If m belongs to S then

E_m denotes the number of elements in S that satisfy exactly m of the t conditions.

$E_1=N(c_1)+N(c_2)+N(c_3)-2[N(c_1c_2)+ N(c_1c_3)+ N(c_2c_3)]+3N(c_1c_2c_3) \quad =S_1-2S_2+3S_3$

$=S_1-C(2,1)S_2+C(3, 2)S_3$

$E_2=N(c_1c_2)+N(c_1c_3)+N(c_2c_3)-3N(c_1c_2c_3)$

$=S_2-3S_3=S_2-C(3, 1)S_3$

$E_3=S_3$

No of conditions =4 it can be represented by

$E_1=S_1-C(2,1)S_2+ C(3, 2)S_3-C(4, 3)S_4$

$E_2=S_2-C(3,1)S_3+ C(4, 2)S_4$

$E_3=S_3-C(4,1)S_4$

$E_4=S_4$

Theorem : For each 1<=m<=t , the number of elements in S that exactly given m of the conditions $c_1, c_2, c_3, \ldots$ C t is give by

$$E_m = S_m - \binom{m+1}{1} S_{m-1} + \binom{m+2}{2} S_{m+2} - \cdots + (-1)^{t-m} \binom{t}{t-m} S_t.$$

Let L_m denotes the number of elements in S that satisfy at least m of the t conditions.

$$L_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} - \cdots + (-1)^{t-m} \binom{t-1}{m-1} S_t.$$

Pigeon hole principles and its application:

The statement of the *Pigeonhole Principle*:

If m pigeons are put into m pigeonholes, there is an empty hole if there's a hole with more than one pigeon. If n > m pigeons are put into m pigeonholes, there's a hole with more than one pigeon.

Example:

Consider a chess board with two of the diagonally opposite corners removed. Is it possible to cover the board with pieces of domino whose size is exactly two board squares?

Solution

No, it's not possible. Two diagonally opposite squares on a chess board are of the same color. Therefore, when these are removed, the number of squares of one color exceeds by 2 the number of squares of another color. However, every piece of domino covers exactly two squares and these are of different colors. Every placement of domino pieces establishes a 1-1 correspondence between the set of white squares and the set of black squares. If the two sets have different number of elements, then, by the Pigeonhole Principle, no 1-1 correspondence between the two sets is possible.

Generalizations of the pigeonhole principle

A generalized version of this principle states that, if *n* discrete objects are to be allocated to *m* containers, then at least one container must hold no fewer than $\lceil n/m \rceil$ objects, where $\lceil x \rceil$ is the ceiling function, denoting the smallest integer larger than or equal to *x*. Similarly, at least one container must hold no more than $\lfloor n/m \rfloor$ objects, where $\lfloor x \rfloor$ is the floor function, denoting the largest integer smaller than or equal to *x*.

A probabilistic generalization of the pigeonhole principle states that if *n* pigeons are randomly put into *m* pigeonholes with uniform probability 1/*m*, then at least one pigeonhole will hold more than one pigeon with probability

$$1 - \frac{(m)_n}{m^n},$$

where $(m)_n$ is the falling factorial $m(m - 1)(m - 2)...(m - n + 1)$. For $n = 0$ and for $n = 1$ (and $m > 0$), that probability is zero; in other words, if there is just one pigeon, there cannot be a conflict. For $n > m$ (more pigeons than pigeonholes) it is one, in which case it coincides with the ordinary pigeonhole principle. But even if the number of pigeons does not exceed the number of pigeonholes ($n \leq m$), due to the random nature of the assignment of pigeons to pigeonholes there is often a substantial chance that clashes will occur. For example, if 2 pigeons are randomly assigned to 4 pigeonholes, there is a 25% chance that at least one pigeonhole will hold more than one pigeon; for 5 pigeons and 10 holes, that probability is 69.76%; and for 10 pigeons and 20 holes it is about 93.45%. If the number of holes stays fixed, there is always a greater probability of a pair when you add more pigeons. This problem is treated at much greater length at birthday paradox.

A further probabilistic generalization is that when a real-valued random variable *X* has a finite mean *E*(*X*), then the probability is nonzero that *X* is greater than or equal to *E*(*X*), and similarly the probability is nonzero that *X* is less than or equal to *E*(*X*). To see that this implies the standard pigeonhole principle, take any fixed arrangement of *n* pigeons into *m* holes and let *X* be the number of pigeons in a hole chosen uniformly at random. The mean of *X* is *n*/*m*, so if there are more pigeons than holes the mean is greater than one. Therefore, *X* is sometimes at least2.

Applications:

The pigeonhole principle arises in computer science. For example, collisions are inevitable in a hash table because the number of possible keys exceeds the number of indices in the array. No hashing algorithm, no matter how clever, can avoid these collisions. This principle also proves that any general-purpose lossless compression algorithm that makes at least one input file smaller will make some other input file larger. (Otherwise, two files would be compressed to the same smaller file and restoring them would be ambiguous.)

5.2.3 Derangements: nothing is in its right place

Derangement means that all numbers are in the wrong positions.

e^{-1}=1-1+(1/2!)-(1/3!)+(1/4!)-(1/5!)+….. =0.36788

Problem : Determine the number of derangements of 1,2,…, 10.

Let c_i be the condition that integer i is in the i-th position. d_{10} can be computed as follows.

$$d_{10} = N(\bar{c}_1\bar{c}_2\bar{c}_3 \cdots \bar{c}_{10}) = 10! - \binom{10}{1}9! + \binom{10}{2}8! - \binom{10}{3}7! + \cdots + \binom{10}{10}0!$$
$$= 10!\left[1 - \binom{10}{1}(9!/10!) + \binom{10}{2}(8!/10!) - \binom{10}{3}(7!/10!) + \cdots + \binom{10}{10}(0!/10!)\right]$$
$$= 10![1 - 1 + (1/2!) - (1/3!) + \cdots + (1/10!)] \doteq (10!)(e^{-1}).$$

The general formula for the Derangements can be represented by

$$d_n = n![1-1+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-....\frac{1}{n!}]$$

=> $$d_n = n!\times e^{-1}$$

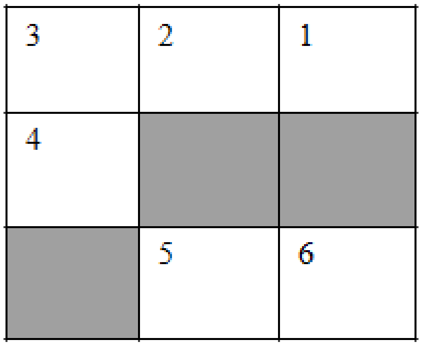
Problem : We have seven books and seven reviewers. Each book needs to be reviewed by two persons. How many ways can we assign the referees?

1. The first week has 7! ways to assign referees.
2. The second week has d_7 ways to assign referees.
3. Totally, we have $7!\times d_7$ ways of possible assignments.

5.2.4 Rook Polynomials

A piece in chess board is called as rook or Castle. No two numbers are in the same order is called as r_k or $r_k(c)$ where C is called as Chessboard

In Figure below, we want to determine the number of ways in which k rooks can be placed on the unshaded squares of this chessboard so that no two of them can take each other—that is, no two of them are in the same row or column of the chessboard. This number is denoted as $r_k(C)$.



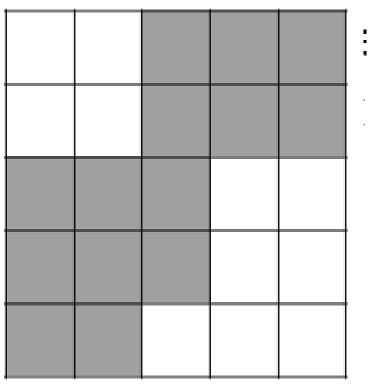
In Fig , we have $r_0=1$, $r_1=6$, $r_2=8$, $r_3=2$ and $r_k=0$ for $k\geq4$.

$r(C, x)=1+6x+8x^2+2x^3$. For each $k\geq0$, the coefficient of x^k is the number of ways we can place k nontaking rooks on chessboard C.

Disjoint subboards :

In below Figure the chessboard contains two disjoint subboards that have no squares in the same column or row of C.

$$r(C, x)=r(C_1, x).\ r(C_2, x).$$



$$r(C_1, x) = 1 + 4x + 2x^2, \qquad r(C_2, x) = 1 + 7x + 10x^2 + 2x^3,$$

$$(C, x) = 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 4x^5 = r(C_1, x) \cdot r(C_2, x).$$

5.2.5 Arrangements with forbidden positions:

Below fig shows a seating arrangement of seating to 4 persons in 5 tables in forbidden places. The shaded square of R_iT_j means R_i will not sit at T_j.

Problems : Determine the number of ways that we can seat these four relatives on un-shaded squares.

Let S be the total number of ways we can place these four relatives, one to a table.

Let c_i be the condition that R_i is seated in a forbidden position but at different tables.

	T_1	T_2	T_3	T_4	T_5
R_1	■	■			
R_2		■			
R_3			■	■	
R_4				■	■

Let r_i be the number of ways in which it is possible to place i nontaking rooks on the shaded chessboard. For all $0 \le i \le 4$, $S_i = r_i(5-i)!$

$R(C, x) = (1+3x+x^2)(1+4x+3x^2)$

$= 1+7x+16x^2+13x^3+3x^4$

$$N(\overline{c_1}\overline{c_2}\overline{c_3}\overline{c_4}) = S_0 - S_1 + S_2 - S_3 + S_4$$
$$= 5! - 7(4!) + 16(3!) - 13(2!) + 3(1!)$$
$$= \sum_{i=0}^{4} (-1)^i r_i (5-i)!$$

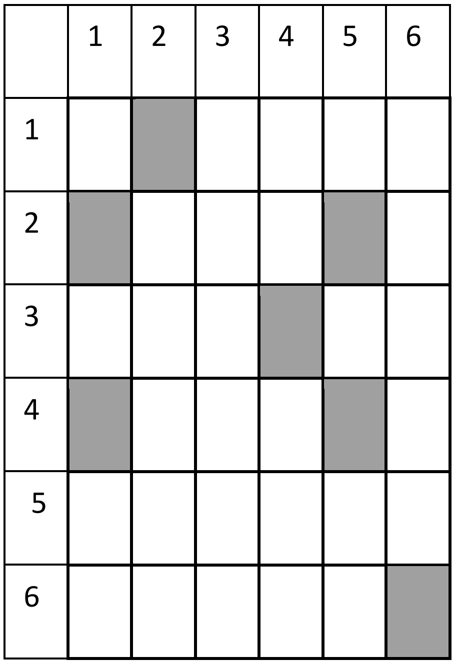
Problem : We roll two dice six times, where one is red die and the other green die.

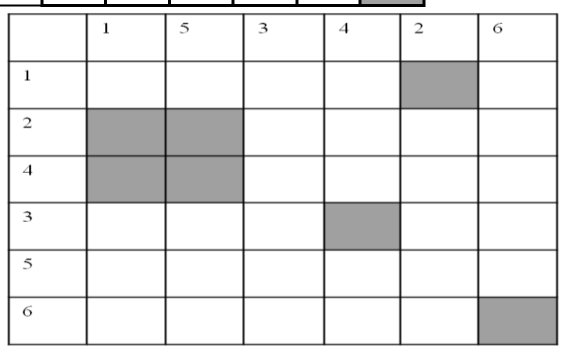
We know the following pairs did not occur: (1, 2), (2, 1), (2, 5), (3, 4), (4, 1), (4, 5) and (6, 6).

What is the probability that we obtain all six values both on red die and green die?

One of solutions is like (1, 1), (2, 3), (4, 4), (3, 2), (5, 6), (6, 5).

$r(C, x) = (1+4x+2x^2)(1+x)^3 = 1+7x+17x^2+19x^3+10x^4+2x^5$, Where c_i denotes that all six values occur on both the red and green dies, but i on the red die is paired with one of the forbidden numbers on the green die. These are represented in below fig's.





$$
\begin{aligned}
(6!)N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4\bar{c}_5\bar{c}_6) &= (6!)\sum_{i=0}^{6}(-1)^i S_i = (6!)\sum_{i=0}^{6}(-1)^i r_i(6-i)! \\
&= 6![6! - 7(5!) + 17(4!) - 19(3!) + 10(2!) - 2(1!) + 0(0!)] \\
&= 6![192] = 138{,}240.
\end{aligned}
$$

Since the sample space consists of all sequences of six ordered pairs selected with repetition from the 29 unshaded squares of the chessboard, the probability of this event is $138.240/(29)^6 \doteq 0.00023$.

5.3 Generating Functions

5.3.1 Introductory Examples

Problem 1: 12 oranges for three children, Grace, Mary, and Frank.

Grace gets at least four, and Mary and Frank gets at least two, but Frank gets no more than five. $(x^4+ x^5+ x^6+ x^7+ x^8)\ (x^2+ x^3+x^4+ x^5+ x^6)(x^2+ x^3+x^4+ x^5)$ Then The coefficient of the Problem is x^{12}

Problem 2 : 12 oranges for three children, Grace, Mary, and Frank.

Grace gets at least four, and Mary and Frank gets at least two, but Frank gets no more than five. $(x^4+ x^5+ x^6+ x^7+ x^8)\ (x^2+ x^3+x^4+ x^5+ x^6)(x^2+ x^3+x^4+ x^5)$

The coefficient of x^{12} is the solution.

Problem 3 : How many nonnegative integer solutions are there for $c_1+c_2+c_3+c_4=25$?

$f(x)=(1+ x^1+ x^2+....+ x^{24}+ x^{25})^{4}$, The coefficient of x^{25} is the solution.

5.3.2 Definition and examples: Calculational Techniques

Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers. The function

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots = \sum_{i=0}^{\infty} a_i x^i$$

is called the *generating function* for the given sequence.

Problem : $(1+x)^n$ is the generating function for the sequence C(n, 0), C(n, 1),…, C(n, n), 0,0,0…

$(1-x^{n+1})/(1-x)$ is the generating function for the sequence 1,1,1,…,1, 0, 0,0…, where the first n+1 terms are 1.

Examples : $1/(1-x)$ is the generating function for the sequence 1,1,1,…,1,….

$1/(1-x)^2$ is the generating function for the sequence 1,2,3,4,….

$x/(1-x)^2$ is the generating function for the sequence 0,1,2,3,….

$(x+1)/(1-x)^3$ is the generating function for the sequence $1^2,2^2,3^2,4^2$,….

$x(x+1)/(1-x)^3$ is the generating function for the sequence $0^2,1^2,2^2,3^2,4^2$,….

The above can be represented by

$$f_0(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

$$f_1(x) = x\frac{d}{dx} f_0(x) = \frac{x}{(1-x)^2}$$

$$= 0 + x + 2x^2 + 3x^3 + \cdots$$

$$f_2(x) = x\frac{d}{dx} f_1(x) = \frac{x^2 + x}{(1-x)^3}$$

$$= 0^2 + 1^2x + 2^2x^2 + 3^2x^3 + \cdots$$

$$f_3(x) = x\frac{d}{dx} f_2(x) = \frac{x^3 + 4x^2 + x}{(1-x)^4}$$

$$= 0^3 + 1^3x + 2^3x^2 + 3^3x^3 + \cdots$$

$$f_4(x) = x\frac{d}{dx} f_3(x) = \frac{x^4 + 11x^3 + 11x^2 + x}{(1-x)^5}$$

$$= 0^4 + 1^4x + 2^4x^2 + 3^4x^3 + \cdots$$

Problem 2 : 1/(1-ax) is the generating function for the sequence a^0,a^1,a^2,a^3,….

Let f(x)=1/(1-x). Then g(x)=f(x)-x^2 is the generating function for the sequence 1,1,0, 1, 1,…,….

Let $a_i=i^2+i$ for $i\geq 0$. Then its generating function is $[x(x+1)/(1-x)^3]+[x/(1-x)^2]=2x/(1-x)^3$

Define C(n, r) for n∈R :

Since when $n\in Z^+$, we have

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)...(n-r+1)}{r!}$$

So for n ∈R we define

$$\binom{n}{r} = \frac{n(n-1)(n-2)...(n-r+1)}{r!}$$

For example, if n is positive, we have

$$\binom{-n}{r} = \frac{(-n)(-n-1)(-n-2)...(-n-r+1)}{r!}$$

$$= \frac{(-1)^r (n)(n+1)...(n+r-1)}{r!}$$

$$= \frac{(-1)^r (n+r-1)!}{(n-1)!r!} = (-1)^r \binom{n+r-1}{r}$$

5.3.3 Partition of integers

p(x) is the number of partitions for x.

For n, the number of 1's is 0 or 1 or 2 or 3…. The power series is $1+x+x^2+x^3+x^4+…$.

For n, the number of 2's can be kept tracked by the power series $1+x^2+x^4+x^6+x^8+…$.

For n, the number of 3's can be kept tracked by the power series $1+x^3+x^6+x^9+x^{12}+…$.

$f(x)=(1+x+x^2+x^3+x^4+…)(1+x^2+x^4+x^6+x^8+x^{10}+…) (1+x^3+x^6+x^9+…) … (1+x^{10}+…)$

$=1/(1-x)\times 1/(1-x^2)\times 1/(1-x^3) \times…\times 1/(1-x^{10})$

At last, we have the following series for p(n) by the coefficient of x^n

$$\prod_{i=1}^{\infty}[1/(1-x^i)]$$

Problem : Find the number of ways an advertising agent can purchase n minutes if the time slots come in blocks of 30, 60, 120 seconds.

Let 30 seconds represent one time unit.

a+2b+4c=2n

$f(x)= (1+x+x^2+x^3+x^4+…) (1+x^2+x^4+x^6+x^8+…)(1+x^4+x^8+x^{12}+…)$

$=1/(1-x) \times 1/(1-x^2) \times 1/(1-x^4)$.

The coefficient of x^{2n} is the answer to the problem.

Problem :

$p_d(n)$ is the number of partitions of a positive integer n into distinct summands.

$P_d(x)=(1+x)(1+x^2)(1+x^3).....$

Problem :

$p_o(n)$ is the number of partitions of a positive integer n into odd summands.

$P_o(x)= (1+x+x^2+x^3+x^4+...) (1+x^3+x^6+x^9+x^{12}+...)(1+x^5+x^{10}+x^{15}+...)...$

$P_o(x)=1/(1-x) \times 1/(1-x^3) \times 1/(1-x^5) \times 1/(1-x^7) \times ...$

$P_d(x)= P_o(x)$

Problem

$p_{oo}(n)$ is the number of partitions of a positive integer n into odd summands and such summands must occur an odd number of times.

$P_{oo}(x)= (1+x+x^3+x^5+x^7+...) (1+x^3+x^9+x^{15}+...)(1+x^5+x^{15}+x^{25}+...)...$

5.3.4 The exponential generating function:

Now for all $0 \le r \le n$,

$$C(n, r) = \frac{n!}{r!(n-r)!} = \left(\frac{1}{r!}\right) P(n, r),$$

where $P(n, r)$ denotes the number of permutations of n objects taken r at a time. So

$$\begin{aligned}(1+x)^n &= C(n, 0) + C(n, 1)x + C(n, 2)x^2 + C(n, 3)x^3 + \cdots + C(n, n)x^n \\ &= P(n, 0) + P(n, 1)x + P(n, 2)\frac{x^2}{2!} + P(n, 3)\frac{x^3}{3!} + \cdots + P(n, n)\frac{x^n}{n!}.\end{aligned}$$

For a sequence $a_0, a_1, a_2, a_3, \ldots$ of real numbers,

$$f(x) = a_0 + a_1 x + a_2\frac{x^2}{2!} + a_3\frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!},$$

is called the *exponential generating function* for the given sequence.

Problem:
In how many ways can four of the letters in ENGINE be arranged?
$f(x)=[1+x+(x^2/2!)]^2[1+x]^2$, and the answer is the coefficient of $x^4/4!$.

In the complete expansion of $f(x)$, the term involving x^4 [and, consequently, $x^4/4!$] is

$$\left(\frac{x^4}{2!\,2!}+\frac{x^4}{2!}+\frac{x^4}{2!}+\frac{x^4}{2!}+\frac{x^4}{2!}+\frac{x^4}{2!}+\frac{x^4}{2!}+x^4\right)$$
$$=\left[\left(\frac{4!}{2!\,2!}\right)+\left(\frac{4!}{2!}\right)+\left(\frac{4!}{2!}\right)+\left(\frac{4!}{2!}\right)+\left(\frac{4!}{2!}\right)+\left(\frac{4!}{2!}\right)+\left(\frac{4!}{2!}\right)+4!\right]\left(\frac{x^4}{4!}\right).$$

Important Series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots.$$

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots.$$

Problem :
We have 48 flags, 12 each of the colors red, white, blue and black. Twelve flags are placed on a vertical pole to show signal.
How many of these use an even number of blue flags and an odd number of black flags?

Solution:

$$f(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^2 \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

Then

$$f(x) = (e^x)^2 \left(\frac{e^x + e^{-x}}{2}\right) \left(\frac{e^x - e^{-x}}{2}\right) = \left(\frac{1}{4}\right) (e^{2x})(e^{2x} - e^{-2x}) = \frac{1}{4}(e^{4x} - 1)$$

$$= \frac{1}{4}\left(\sum_{i=0}^{\infty} \frac{(4x)^i}{i!} - 1\right) = \left(\frac{1}{4}\right) \sum_{i=1}^{\infty} \frac{(4x)^i}{i!},$$

the coefficient of $x^{12}/12!$ in $f(x)$ yields $(1/4)(4^{12}) = 4^{11}$ signals made up of 12 flags with an even number of blue flags and an odd number of black flags.

5.3.5 The Summation Operator:

Let $f(x)=a_0+a_1x+a_2x^2+a_3x^3+\ldots$. Then $f(x)/(1-x)$ generate the sequence of a_0, a_0+a_1, $a_0+a_1+a_2$, $a_0+a_1+a_2+a_3$,…

So we refer to 1/(1-x) as the summation operator.

$$\frac{f(x)}{1-x} = f(x) \cdot \frac{1}{1-x} = [a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots][1 + x + x^2 + x^3 + \cdots]$$

$$= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + (a_0 + a_1 + a_2 + a_3)x^3 + \cdots,$$

Problem:

1/(1-x) is the generating function for the sequence 1, 1, 1, 1, 1,…
[1/(1-x)]× [1/(1-x)] is the generating function for the sequence 1,2,3,4,5,…
$x+x^2$ is the generating function for the sequence 0, 1, 1, 0, 0, 0,…
$(x+x^2)/(1-x)$ is the generating function for the sequence 0, 1, 2, 2, 2, 2, …
$(x+x^2)/(1-x)^2$ is the generating function for the sequence 0, 1, 3, 5, 7, 9, 11, …
$(x+x^2)/(1-x)^3$ is the generating function for the sequence 0, 1, 4, 9, 16, 25, 36, …

Problem:

$g(x)= 1/(1-x)=1+x+x^2+x^3+x^4+...$

$q(x)=dg(x)/dx=1/(1-x)^2=1+2x+3x^2+4x^3+....$

$r(x)=xq(x)=x/(1-x)^2 = x+2x^2+3x^3+4x^4+....$

$xdr(x)/dx=(1+x)/(1-x)^3= x+2^2x^2+3^2x^3+4^2x^4+.$

$x(1+x)/(1-x)^4 = x+(1^2+2^2)x^2+(1^2+2^2+3^2)x^3+(1^2+2^2+3^2+4^2)x^4+....$