LECTURE NOTES

on

(15A05302) DISCRETE MATHEMATICS

II B.Tech – I Semester (JNTUA-R15)

Dr K Sreenivasulu

M.Tech;Ph.D;

Professor of Computer Science & Engineering Department.

GPCET
Pioneering Innovative Education

Department of Computer Science & Engineering

G.Pullaiah College of Engineering & Technology

Nandikotkur Road-VENKAYAPALLI

KURNOOL-518002

JAWAHARLAL NEHRU TECHNOLOGICAL UNIVERSITY ANANTAPUR

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(15A05302) DISCRETE MATHEMATICS

II Year B.Tech.I Sem.

Course Objectives

- Understand the methods of discrete mathematics such as proofs, counting principles, number theory, logic and set theory.
- Understand the concepts of graph theory, binomial theorem, and generating function in analysis of various computer science applications.

Course Outcomes Able to apply mathematical concepts and logical reasoning to solve problems in different fields of Computer science and information technology.

- Able to apply the concepts in courses like Computer Organization, DBMS, Analysis of Algorithms, Theoretical Computer Science, Cryptography, Artificial Intelligence

UNIT I:

Mathematical Logic:
Introduction, Connectives, Normal Forms, The theory of Inference for the Statement Calculus,
The Predicate Calculus, Inference Theory of Predicate Calculus.

UNIT II:

SET Theory:
Basic concepts of Set Theory, Representation of Discrete structures, Relations and Ordering, Functions, Recursion.

UNIT III:
Algebraic Structures:

Algebraic Systems: Examples and General Properties, Semi groups and Monoids, Polish expressions and their compilation, Groups: Definitions and Examples, Subgroups and Homomorphism's, Group Codes.

Lattices and Boolean algebra:
Lattices and Partially Ordered sets, Boolean algebra.

UNIT IV:

An Introduction to Graph Theory:

Definitions and Examples, Sub graphs, complements, Graph Isomorphism, Vertex Degree: Euler Trails and Circuits, Planar Graphs, Hamilton Paths and Cycles, Graph Coloring and Chromatic Polynomials

Trees:

Definitions, Properties, Examples, Rooted Trees, Trees and Sorting, Weighted trees and Prefix Codes, Biconnected Components and Articulation Points

UNIT V:

Fundamental Principles of Counting:

The rules of Sum and Product, Permutations, Combinations: The Binomial Theorem, Combinations with Repetition

The Principle of Inclusion and Exclusion:

The Principle of Inclusion and Exclusion, Generalizations of Principle, Derangements: Nothing is in Its Right Place, Rook Polynomials, Arrangements with Forbidden Positions

Generating Functions:

Introductory Examples, Definitions and Examples: Calculation Techniques, Partitions of Integers, The Exponential Generating Functions, The Summation Operator.

TEXT BOOKS:

1. “Discrete Mathematical Structures with Applications to Computer Science”, J.P. Tremblay and R. Manohar, McGraw Hill Education,2015.

2. “Discrete and Combinatorial Mathematics, an Applied Introduction”, Ralph P. Grimaldi and B.V.Ramana, Pearson, 5th Edition, 2016.

REFERENCE BOOKS:

1. Graph Theory with Applications to Engineering by NARSINGH DEO, PHI.
2. Discrete Mathematics by R.K.Bishtand H.S. Dhami, Oxford Higher Education.
3. Discrete Mathematics theory and Applications by D.S.Malik and M.K.Sen, Cenegage Learning.
4. Elements of Discrete Mathematics, A computer Oriented approach by C L Liu and D P Mohapatra, MC GRAW HILL Education.
5. Discrete Mathematics for Computer scientists and Mathematicians by JOE L.Mott, Abraham Kandel and Theodore P.Baker, Pearson ,2nd Edition

UNIT – I

Mathematical Logic:

Introduction, Connectives, Normal Forms, The theory of Inference for the Statement Calculus, The Predicate Calculus, Inference Theory of Predicate Calculus.

1. MATHEMATICAL LOGIC:

INTRODUCTION

1.1 Statements and notations:

A proposition or statement is a declarative sentence that is either true or false (but not both). For instance, the following are propositions:

1. "Paris is in France" (true), "London is in Denmark"(false),
2. "2 < 4" (true), "4 = 7 (false)".

However the following are not propositions:

1. "what is your name?" (this is a question),
2. "do your homework" (this is a command),
3. "this sentence is false"(neither true nor false),
4. "x is an even number" (it depends on what x represents),
5. "Socrates" (it is not even a sentence).

The truth or falsehood of a proposition is called its truth value.

Notations:

These statements are represented by either capital or small letters i.e P,Q or p,q….. are called as proposition variables.

1.2 Connectives:

Connectives are used for making compound propositions i.c combination of 2 or more variables. They are

Connective	Representation	Meaning
Negation	$\neg p$	Not P
Conjunction	$p \wedge q$	P and Q
Disjunction	$p \vee q$	P or Q
Conditional	$p \rightarrow q$	If… then
Bi-conditional	$p \leftrightarrow q$	If and only if

Truth Table:

A truth table is table that shows the truth values of a compound statement for all possible cases. They are represented as

1. **Logical negation**

Logical negation is an operation on one logical value, typically the value of a proposition that produces a value of *true* if its operand is false and a value of *false* if its operand is true.
The truth table for **NOT p** (also written as **¬p** or **~p**) is as follows:

P	Q	~P	~Q
T	T	F	F
T	F	F	T
F	T	T	F
T	F	T	T

2. **Logical Disjunction**

Logical conjunction is an operation on two logical values, typically the values of two propositions, that produces a value of *true* if both of its operands are true.
The truth table for **p AND q** (also written as **p q**, **p & q**, or **p q**) is as follows:

P	Q	***p* ∧ *q***
T	T	T
T	F	F
F	T	F
F	F	F

3. Legal Disjunction

Exclusive disjunction is an operation on two logical values, typically the values of two propositions, that produces a value of *true* if one but not both of its operands is true. The truth table for **p XOR q** (also written as **p q**, or) is as follows:

P	Q	***p* ∨ *q***
T	T	T
T	F	T
F	T	T
F	F	F

4. Legal Conditional

Logical implication and the material conditional are both associated with an operation on two logical values, typically the values of two propositions, that produces a value of *false* just in the singular case the first operand is true and the second operand is false. The truth table associated with the material conditional **if p then q** (symbolized as **p → q**) and the logical implication **p implies q** (symbolized as **p→q**) is as follows:

P	Q	p → q
T	T	T
T	F	F

F	T	T
F	F	T

5. Bi-conditional

The truth table associated with the bi-conditional if and only if (symbolized as **p ↔ q) is as follows.**

P	Q	p ↔ q
T	T	T
T	F	F
F	T	F
F	F	T

CONVERSE , INVERSE, CONTRAPOSITIVE STATEMENTS:

Converse:

The statements obtained by interchanging the hypothesis is called converse statement can be represented as If P→Q then the converse statement is Q→P .

Inverse:

The statement obtained by taking the negative of the hypothesis & conclusion in the conditional statement is called as inverse of the given statement.

Ex: if P→Q the inverse statement is ~P →~Q

Contrapositive Statement

The statement obtained by taking the negation of the converse of the original statement is called the Contrapositive statement.

Ex: if P→Q the Contrapositive statement is ~Q →~P

1.3 Well-formed formulas:

Not all strings can represent propositions of the predicate logic. Those which produce a proposition when their symbols are interpreted must follow the rules given below, and they are called wffs (well-formed formulas) of the first order predicate logic.

Rules for constructing Well-formed formulas:

A predicate name followed by a list of variables such as P(x, y), where P is predicate name, and x and y are variables, is called an atomic formula.

A well formed formula of predicate calculus is obtained by using the following rules.

1. An atomic formula is a wff.
2. If A is a wff, then ~A is also a wff.
3. If A and B are wffs, then (A V B), (A ∧ B),(A→B), (A↔B) are also Well-Formed formulas.
4. If A is a wff and x is a any variable, then (x)A and ($x)A are wffs.
5. Only those formulas obtained by using (1) to (4) are wffs.

Since we will be concerned with only wffs, we shall use the term formulas for wff. We shall follow the same conventions regarding the use of parentheses as was done in the case of statement formulas.

Wffs are constructed using the following rules:

1. *True* and *False* are wffs.
2. Each propositional constant (i.e. specific proposition), and each propositional variable (i.e. a variable representing propositions) are wffs.
3. Each atomic formula (i.e. a specific predicate with variables) is a wff.
4. If *A, B,* and *C* are wffs, then so are ~*A*, (*A* V *B*), (*A* ∧ *B)*, *(A* → *B),* and *(A* ↔ *B)*.
5. If *x* is a variable (representing objects of the universe of discourse), and *A* is a wff, then so are ∀*x A* and ∃*x A* .

Tautology , Contradiction and Contingency:

Tautology:

A proposition is said to be a tautology if its truth value is T for any assignment of truth values to its components.

Example: The proposition p ¬p is a tautology.

Contradiction:

A proposition is said to be a contradiction if its truth value is F for any assignment of truth values to its components.

Example: The proposition p ¬p is a contradiction.

Contingency

A proposition that is neither a tautology nor a contradiction is called a contingency.

1.4 Normal forms:

Let A(P1, P2, P3, …, Pn) be a statement formula where P1, P2, P3, …, Pn are the atomic variables. If A has truth value T for all possible assignments of the truth values to the variables P1, P2, P3, …, Pn , then A is said to be a tautology. If A has truth value F, then A is said to be identically false or a contradiction. Such Truth Table Contains 2^n possible rows for n number of variables

Construction of truth table involves a finite number of steps and such a decision problem in the statement calculus always has a solution.

Construction of truth table may not be practical, even with the aid of a computer, to overcome this problem consider other procedures known as Normal Forms.

They are

1. Disjunctive Normal Forms

2. Conjunctive normal Forms

Disjunctive Normal Forms

A product of the variables and their negations in a formula is called an elementary product. A sum of the variables and their negations is called an elementary sum. That is, a sum of elementary products is called a disjunctive normal form of the given formula.

Examples:

Ex 1: Obtain DNF for the following Statements. $P \wedge (P \rightarrow Q)$

Sol: $P \wedge (P \rightarrow Q)$

$P \wedge (\sim P \vee Q)$

$(P \wedge \sim P) \vee (P \wedge Q)$

Conjunctive Normal Forms

A formula which is equivalent to a given formula and which consists of a product of elementary sum is called as conjunctive normal form of a given formula.

Examples:

Ex1: Obtain the CNF for the following compound statement $P \wedge (P \rightarrow Q)$

Sol: $P \wedge (P \rightarrow Q)$

$P \wedge (\sim P \vee Q)$

Principle Conjunctive Normal Form:

If a compound statement is given , we can write equivalent proposition as disjunction of minterms , which is called as PCNF where minterms are $P \wedge Q$, $\sim P \wedge Q$, $P \wedge \sim Q$, $\sim P \wedge \sim Q$

Ex: $(P \wedge Q) \vee (\sim P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$

Principle Disjunctive Normal forms:

If a compound statement is given , we can write equivalent proposition as Conjunctive of maxterms , which is called as PCNF where maxterms are $P \vee Q$, $\sim P \vee Q$, $P \vee \sim Q$, $\sim P \vee \sim Q$

Ex: $(P \vee Q) \wedge (\sim P \vee Q) \wedge (P \vee \sim Q) \wedge (\sim P \vee \sim Q)$

1.5 The theory of Inference for the Statement Calculus

The main Problem in the logic is the investigation of the process of reasoning. A certain set of statements is assumed and from the set and other statements are derived by logical reasoning.

Arguments: An argument is a list of propositions P1,P2,P3,P4… Pn followed by a preposition Q is called the conclusion.

An argument is said to be valid if and only if the conjunction of the premises implies the conclusion i.e the premises are True then conclusion is True

i.e $P1 \wedge P2 \wedge P3 \wedge P4 \leftrightarrow Q$

Where P1 , P2 ,P3 … are called as Premises and Q is called as Conclusion.

The arguments are verified by using 2 Methods

1. By Using Truth Table
2. Without using truth table

Truth Table

Arguments can be verified by using truth table by using the following procedure.

1. Identify the premises and conclusion.
2. Construct the truth table for premises and conclusion
3. Find the rows in which all the rows are true.
4. From the table determine the conclusion is T , Then it is a Valid arguments otherwise they are invalid.

Problem: 1

Verify the conclusion C: ~p follows H1: ~q ; H2: p→q using Truth table.

P	q	C	H1	H2
T	T	F	T	F
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

From the table T is both for Premises H1, H2 and Conclusion C , So the arguments are Valid.

Rules of inference:

The two rules of inference are called rules P and T.

(i) Rule P: A premise may be introduced at any point in the derivation.

(ii) Rule T: A formula S may be introduced in a derivation if s is tautologically implied by any one or more of the preceding formulas in the derivation.

Before preceding the actual process of derivation, some important list of implications and equivalences are given in the following tables.

Implications

I1 $P \wedge Q \Rightarrow P$ } Simplification

I2 $P \wedge q \Rightarrow Q$

I3 $P \Rightarrow P \vee Q$ addition

I4 Q=>PVQ

I5∼ P =>P→ Q

I6 Q =>P→Q

I7∼ (P→Q)=> P

I8 ∼ (P → Q)=> ∼ Q

I9 P, Q => P ∧Q

I10 ∼ P, PVQ=>Q (disjunctivesyllogism)

I11 P, P→ Q=>Q (modus ponens)

I12 ∼ Q, P → Q=>∼ P (modustollens)

I13 P → Q, Q → R => P→R
(hypotheticalsyllogism)

I14 P V Q, P → Q, Q → R=>R (dilemma)

Equivalences

E1 ∼ ∼ P <=>P

E2 P ∧ Q <=>Q∧P } Commutative laws

E3 P V Q <=>Q V P

E4 (P ∧ Q) ∧ R <=>P ∧ (Q ∧ R) } Associative laws

E5 (P V Q) V R <=>PV (Q V R)

E6 P ∧ (Q V R) <=>(P ∧ Q) V (P ∧ R) } Distributive laws

E7 P V (Q ∧ R) <=>(P V Q) ∧ (PVR)

E8 ∼ (P ∧ Q) <=>∼ P V∼ Q

E9 ∼ (P V Q) <=>∼ P ∧ ∼ Q } De Morgan's laws

E10 P V P =>P

E11 P ∧ P =>P

E12 R V (P ∧ ∼ P) <=>R

E13 R ∧ (P V ∼ P) <=>R

E14 R V (P V ∼ P) <=>T

E15 R ∧ (P ∧ ∼ P) <=>F

E16 $P \to Q \iff \sim P \vee Q$

E17 $\sim (P \to Q) \iff P \wedge \sim Q$

E18 $P \to Q \iff \sim Q \to \sim P$

E19 $P \to (Q \to R) \iff (P \wedge Q) \to R$

E20 ~ (PD Q) <=>P D 7Q

E21 $P \iff Q \iff (P \to Q) \wedge (Q \to P)$

E22 $(P \iff Q) \iff (P \wedge Q) \vee (\sim P \wedge \sim Q)$

Problems

Example 1.Show that R is logically derived from $P \to Q$, $Q \to R$, and P

Solution.	{1}	(1)	$P \to Q$	Rule P
	{2}	(2)	P	Rule P
	{1, 2}	(3)	Q	Rule (1), (2) and I11
	{4}	(4)	$Q \to R$	Rule P
	{1, 2, 4}	(5)	R	Rule (3), (4) and I11.

Example 2.Show that S V R tautologically implied by $(P \vee Q) \wedge (P \to R) \wedge (Q \to S)$.

Solution .	{1}	(1)	P V Q	Rule P
	{1}	(2)	$7P \to Q$	T, (1), E1 and E16
	{3}	(3)	$Q \to S$	P
	{1, 3}	(4)	$7P \to S$	T, (2), (3), and I13
	{1, 3}	(5)	$7S \to P$	T, (4), E13 and E1
	{6}	(6)	$P \to R$	P
	{1, 3, 6}	(7)	$7S \to R$	T, (5), (6), and I13
	{1, 3, 6)	(8)	S V R	T, (7), E16 and E1

Example 3. Show that Q, $P \to Q$ => 7P

Solution .	{1}	(1)	$P \to Q$	Rule P
	{1}	(2)	$7P \to 7Q$	T, and E 18
	{3}	(3)	7Q	P
	{1, 3}	(4)	7P	T, (2), (3), and I11 .

Example 4 .Prove that R ^(P V Q) is a valid conclusion from the premises PVQ , Q → R, P → M and 7M.

Solution .{1}	(1)	P → M	P
{2}	(2)	7M	P
{1, 2}	(3)	7P	T, (1), (2), and I12
{4}	(4)	P V Q	P
{1, 2 , 4}	(5)	Q	T, (3), (4), and I10.
{6}	(6)	Q → R	P
{1, 2, 4, 6}	(7)	R	T, (5), (6) and I11
{1, 2, 4, 6}	(8)	R ^ (PVQ)	T, (4), (7), and I9.

There is a third inference rule, known as rule CP or rule of *conditional proof*.

Rule CP: If we can derives s from R and a set of premises , then we can derive R → S from the set of premises alone.

Note. 1. Rule CP follows from the equivalence E10 which states that (P^ R) → S óP → (R →S).

2. Let P denote the conjunction of the set of premises and let R be any formula
3. The above equivalence states that if R is included as an additional premise and S is derived from P ^ R then R → S can be derived from the premises P alone.
4. Rule CP is also called the *deduction theorem* and is generally used if the conclusion is of the form R → S. In such cases, R is taken as an additional premise and S is derived from the given premises and R.

Example 5 .Show that R → S can be derived from the premises P → (Q → S), 7R V P , and Q.

Solution	{1}	(1) 7R V P	P
	{2}	(2) R	P, assumed premise
	{1, 2}	(3) P	T, (1), (2), and I10
	{4}	(4) P → (Q → S)	P
	{1, 2, 4}	(5) Q →S	T, (3), (4), and I11
	{6}	(6) Q	P

{1, 2, 4, 6}	(7) S	T, (5), (6), and I11
{1, 4, 6}	(8) R → S	CP.

Example 6.Show that P → S can be derived from the premises, 7P V Q, 7Q V R, and R → S .

Solution.

{1}	(1) 7P V Q	P
{2}	(2) P	P, assumed premise
{1, 2}	(3) Q	T, (1), (2) and I11
{4}	(4) 7Q V R	P
{1, 2, 4}	(5) R	T, (3), (4) and I11
{6}	(6) R → S	P
{1, 2, 4, 6}	(7) S	T, (5), (6) and I11
{2, 7}	(8) P → S	CP

Example 7. " If there was a ball game , then traveling was difficult. If they arrived on time, then traveling was not difficult. They arrived on time. Therefore, there was no ball game". Show that these statements constitute a valid argument.

Solution. Let P: There was a ball game

Q: Traveling was difficult. R:
They arrived on time.

Given premises are: P → Q, R → 7Q and R conclusion is: 7P

1}	(1) P → Q	P
{2}	(2) R → 7Q	P
{3}	(3) R	P
{2, 3}	(4) 7Q	T, (2), (3), and I11
{1, 2, 3}	(5) 7P	T, (2), (4) and I12

1.5 PREDICATE CALCULUS

Predicate Logic:

A predicate or propositional function is a statement containing variables. For instance "x + 2 = 7", "X is American", "x < y", "p is a prime number" are predicates. The truth value of the predicate depends on the value assigned to its variables. For instance if we replace x with 1 in the predicate "x + 2 = 7" we obtain "1 +

2 = 7”, which is false, but if we replace it with 5 we get “5 + 2 = 7”, which is true. We represent a predicate by a letter followed by the variables enclosed between parentheses: P (x), Q(x, y), etc. An example for P (x) is a value of x for which P (x) is true. A counterexample is a value of x for which P (x) is false. So, 5 is an example for “x + 2 = 7”, while 1is a counter example.

Each variable in a predicate is assumed to belong to a universe (or domain) of discourse, for instance in the predicate “n is an odd integer” ’n’ represents an integer, so the universe of discourse of n is the set of all integers. In “X is American” we may assume that X is a human being, so in this case the universe of discourse is the set of all human beings.

Quantifiers.

Given a predicate P (x), the statement “for some x, P (x)” (or “there is some x such that p(x)”), represented “ x P (x)”, has a definite truth value, so it is a proposition in the usual sense. For instance if P (x) is “x + 2 = 7” with the integers as universe of discourse, then x P (x) is true, since there is indeed an integer, namely 5, such that P (5) is a true statement. However, if Q(x) is “2x = 7” and the universe of discourse is still the integers, then x Q(x) is false. On the other hand, x Q(x) would be true if we extend the universe of discourse to the rational numbers. The symbol is called the existential quantifier.

Analogously, the sentence “for all x, P (x)”—also “for any x, P (x)”, “for every x, P (x)”, “for each x, P (x)”—, represented “ x P (x)”, has a definite truth value. For instance, if P (x) is “x + 2 = 7” and the 1 Usually all variables occurring in predicates along a reasoning are supposed to belong to the same universe of discourse, but in some situations (as in the so called many-sorted logics) it is possible to use di erent kinds of variables to represent di erent types of objects belonging to di erent universes of discourse. For instance in the predicate “s is a string of length n” the variable s represents a string, while n represents a natural number, so the universe of discourse of s is the set of all strings, while the universe of discourse of n is the set of natural numbers. universe of discourse is the integers, then x P (x) is false. However if Q(x) represents “(x + 1) = x + 2x + 1” then x Q(x) is true. The the symbol is called the universal quantifier. In predicates with more than one variable it is possible to use several quantifiers at the same time, for instance x y z P (x, y , z), meaning “for all x and all y there is some z such that P (x, y, z)”.Note that in general the existential and universal quantifiers cannot be swapped, i.e., in general x y P (x, y) means something di erent from y x P (x, y). For instance if x and y represent human beings and P (x, y) represents “x is a friend of y”, then x y P (x, y) means that very body is a friend of someone, but y x P (x, y) means that there is someone such that everybody is his or her friend.

A predicate can be partially quantified, e.g. x y P (x, y , z, t). The variables quantified (x and y in the example) are called bound variables, and the rest (z and t in the example) are called free variables. A partially quantified predicate is still a predicate, but depending on fewer variables.

Generalized De Morgan Laws for Logic.

If x P (x) is false then there is no value of x for which P (x) is true, or in other words, P (x) is always false. Hence ¬ x P (x) = x ¬P (x) .

On the other hand, if x P (x) is false then it is not true that for every x, P (x) holds, hence for some x, P (x) must be false.

Thus: ¬ x P (x) = x ¬P (x) .

This two rules can be applied in successive steps to find the negation of a more complex quantified statement, for instance:

¬ x y p(x, y) = x¬ y P (x, y) = x y ¬P (x, y) .

Exercise : Write formally the statement "for every real number there is a greater real number". Write the negation of that statement.

Answer : The statement is: x y (x < y) (the universe of discourse is the real numbers). Its negation is: x y ¬(x < y), i.e., x y (x < y).

(Note that among real numbers x < y is equivalent to x = y, but formally they are different predicates.)