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Department of Electrical and Electronics Engineering

Bridge Course
On
Digital Signal Processing

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1. INTRODUCTION:

A **SIGNAL** is defined as any physical quantity that changes with time, distance, speed, position, pressure, temperature or some other quantity. A **SIGNAL** is physical quantity that consists of many sinusoidal of different amplitudes and frequencies.

$$\text{Ex}: \; x(t) = 10t$$

$$x(t) = 5x^2 + 20xy + 30y$$

A **System** is a physical device that performs an operations or processing on a signal.

Ex: Filter or Amplifier.

Signal Processing:

It is the analysis, interpretation, and manipulation of signals like sound, images time-varying measurement values and sensor data etc…

For example biological data such as electrocardiograms, control system signals, telecommunication transmission signals such as radio signals, and many others.

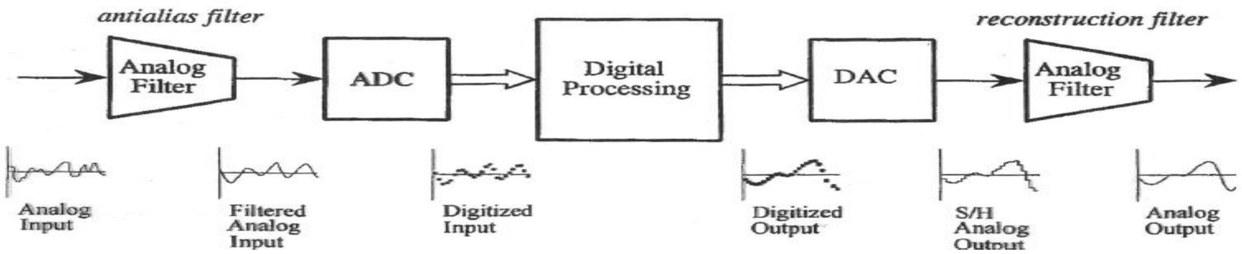
Need of Signal Processing:

When a signal is transmitted from one point to another there is every possibility of contamination /deformation of the signal by external noise. So to retrieve the original signal at the receiver suitable filters are to be used. i.e the signal is processed to obtain the pure signal.

CLASSIFICATION OF SIGNAL PROCESSING

1) **ASP (Analog signal Processing)** : If the input signal given to the system is analog then system does analog signal processing. Ex Resistor, capacitor or Inductor, OP-AMP etc.

2) **DSP (Digital signal Processing):** If the input signal given to the system is digital then system does digital signal processing. Ex Digital Computer, Digital Logic Circuits etc. The digital signal processor consists of anti-aliasing filter, analog to digital converter (ADC), a digital filter represented by the transfer function H(z) called the Anti aliasing Filter, a digital to analog converter and a reconstruction filter.



1. **Anti-aliasing Filter:**

   It ensures that analog input signal does not contain frequency components higher than half of the sampling frequency (to obey the sampling theorem). It eliminates the low frequency harmonic components.

2. **Sample and Hold:**

   It holds a sampled analog value for a short time while the A/D converts and interprets the value as a digital

3. **A/D:**

   It converts a sampled data signal value into a digital number, in part, through quantization of the amplitude

4. **D/A:**

   It converts a digital signal into a \staircase"-like signal(analog signal)

5. **Reconstruction Filter**:

   It converts a \staircase"-like signal into an analog signal through low pass filtering similar to the type used for anti-aliasing.

Most of the signals generated are analog in nature. Hence these signals are converted to digital form by the analog to digital converter. Thus AD Converter generates an array of samples and gives it to the digital signal processor. This array of samples or sequence of samples is the digital equivalent of input analog signal. The DSP performs signal processing operations like filtering, multiplication, transformation or amplification etc operations over this digital signal. The digital output signal from the DSP is given to the DAC.

ADVANTAGES OF DSP OVER ASP

1. Physical size of analog systems are quite large while digital processors are more compact and light in weight.
2. Analog systems are less accurate because of component tolerance ex R, L, C and active components. Digital components are less sensitive to the environmental changes, noise and disturbances.
3. Digital system are most flexible as software programs & control programs can be easily modified.
4. Digital signal can be stores on digital hard disk, floppy disk or magnetic tapes. Hence becomes transportable. Thus easy and lasting storage capacity.
5. Digital processing can be done offline.
6. Mathematical signal processing algorithm can be routinely implemented on digital signal processing systems. Digital controllers are capable of performing complex computation with constant accuracy at high speed.
7. Digital signal processing systems are upgradeable since that are software controlled.
8. Possibility of sharing DSP processor between several tasks.
9. The cost of microprocessors, controllers and DSP processors are continuously going down. For some complex control functions, it is not practically feasible to construct analog controllers.

10. Single chip microprocessors, controllers and DSP processors are more versatile and powerful.

Disadvantages of DSP over ASP

1. Additional complexity (A/D & D/A Converters)
2. Limit in frequency. High speed AD converters are difficult to achieve in practice. In high frequency applications DSP are not preferred.

CLASSIFICATION OF SIGNALS

1. Single channel and Multi-channel signals
2. Single dimensional and Multi-dimensional signals
3. Continuous time and Discrete time signals.
4. Continuous valued and discrete valued signals.
5. Analog and digital signals.
6. Deterministic and Random signals
7. Periodic signal and Non-periodic signal
8. Symmetrical(even) and Anti-Symmetrical(odd) signal
9. Energy and Power signal

1) Single channel and Multi-channel signals

If signal is generated from single sensor or source it is called as single channel signal. If the signals are generated from multiple sensors or multiple sources or multiple signals are generated from same source called as Multi-channel signal. Example ECG signals. Multi-channel signal will be the vector sum of signals generated from multiple sources.

2) Single Dimensional (1-D) and Multi-Dimensional signals (M-D)

If signal is a function of one independent variable it is called as single dimensional signal like speech signal and if signal is function of M independent variables called as Multi-dimensional signals. Gray scale level of image or Intensity at particular pixel on black and white TV are examples of M-D signals.

3) Continuous time and Discrete time signals.

Sr No	Continuous Time (CTS)	Discrete time (DTS)
1	This signal can be defined at any time instance & they can take all values in the continuous interval(a, b) where a can be -∞ & b can be ∞	This signal can be defined only at certain specific values of time. These time instance need not be equidistant but in practice they are usually takes at equally spaced intervals.
2	These are described by differential equations.	These are described by difference equation.
3	This signal is denoted by x(t).	These signals are denoted by x(n) or notation x(nT) can also be used.
4	The speed control of a dc motor using a techogenerator feedback or Sine or exponential waveforms.	Microprocessors and computer based systems uses discrete time signals.

4) Continuous valued and Discrete Valued signals.

Sr No	Continuous Valued	Discrete Valued
1	If a signal takes on all possible values on a finite or infinite range, it is said to be continuous valued signal.	If signal takes values from a finite set of possible values, it is said to be discrete valued signal.
2	Continuous Valued and continuous time signals are basically analog signals.	Discrete time signal with set of discrete amplitude are called digital signal.

5) Analog and digital signal

Sr No	Analog signal	Digital signal
1	These are basically continuous time & continuous amplitude signals.	These are basically discrete time signals & discrete amplitude signals. These signals are basically obtained by sampling & quantization process.
2	ECG signals, Speech signal, Television signal etc. All the signals generated from various sources in nature are analog.	All signal representation in computers and digital signal processors are digital.

Note: Digital signals (**DISCRETE TIME & DISCRETE AMPLITUDE**) are obtained by sampling the **ANALOG** signal at discrete instants of time, obtaining **DISCRETE TIME** signals and then by quantizing its values to a set of discrete values & thus generating **DISCRETE AMPLITUDE** signals.

Sampling process takes place on x axis at regular intervals & quantization process takes place along y axis. Quantization process is also called as rounding or truncating or approximation process.

6) Deterministic and Random signals

Sr No	Deterministic signals	Random signals
1	Deterministic signals can be represented or described by a mathematical equation or lookup table.	Random signals that cannot be represented or described by a mathematical equation or lookup table.
2	Deterministic signals are preferable because for analysis and processing of signals we can use mathematical model of the signal.	Not Preferable. The random signals can be described with the help of their statistical properties.
3	The value of the deterministic signal can be evaluated at time (past, present or future) without certainty.	The value of the random signal can not be evaluated at any instant of time.
4	Example Sine or exponential waveforms.	Example Noise signal or Speech signal

7) Periodic signal and Non-Periodic signal

The signal x(n) is said to be periodic if $x(N+n) = x(n)$ for all n where N is the fundamental period of the signal. If the signal does not satisfy above property called as Non-Periodic signals. Discrete time signal is periodic if its frequency can be expressed as a ratio of two integers. $f = \frac{k}{N}$ where k is integer constant.

8) Symmetrical (Even) and Anti-Symmetrical (odd) signal

A signal is called as symmetrical (even) if $x(n) = x(-n)$ and if $x(-n) = x(n)$ then signal is odd.

$x_1(n) = \cos(\omega n)$ and $x_2(n) = \sin(\omega n)$ are good examples of even & odd signals

9) Energy signal and Power signal

Discrete time signals are also classified as finite energy or finite average power signals.

The energy of a discrete time signal x(n) is given by

$$E = \sum_{n=-\infty}^{\infty} \left| X^2(n) \right|$$

The average power for a discrete time signal x(n) is defined as

$$P = \lim_{N \to \infty} \frac{1}{(2N+1)} \sum_{n=-\infty}^{\infty} \left| X^2(n) \right|$$

If Energy is finite and power is zero for x(n) then x(n) is an energy signal. If power is finite and energy is infinite then x(n) is power signal. There are some signals which are neither energy nor a power signal.

A signal is referred to as an energy signal, if and only if the total energy of the signal satisfies the condition $0 < E < \infty$. On the other hand, it is referred to as a power signal, if and only if the average power of the signal satisfies the condition $0 < P < \infty$.

DISCRETE TIME SIGNALS AND SYSTEM

There are three ways to represent discrete time signals.

1) **Functional Representation**

$$x(n) = \begin{cases} 4 \text{ for } n = 1,3 \\ -2 \text{ for } n = 2 \\ 0 \text{ else where} \end{cases}$$

2) **Tabular method of representation**

n	-4	-3	-2	-1	0	1	2	3	4
X(n)	0	0	0	0	4	1	3	2	1

3) **Sequence Representation**

$$X(n) = \{0, 4, -2, \underset{\substack{\uparrow \\ n=0}}{4}, 0, \ldots\ldots..\}$$

STANDARD SIGNAL SEQUENCES

1) **Unit sample signal (Unit impulse signal)**

$$\delta(n) = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases} \qquad \text{i.e } \delta(n)=\{1\}$$

2) **Unit step signal**

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

3) **Unit ramp signal**

$$u_r(n) = \begin{cases} n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

4) **Exponential signal**

$$x(n) = a^n = \left(re^{j\phi}\right)^n = r^n e^{j\phi n} = r^n\left(\cos\phi n + j\sin\phi n\right)$$

5) **Sinusoidal waveform**

$$x(n) = A.\text{Sin}(\omega n)$$

Properties of Discrete Time Signals:

1) Shifting: signal x(n) can be shifted in time. We can delay the sequence or advance the sequence. This is done by replacing integer n by n-k where k is integer. If k is positive signal is delayed in time by k samples (Arrow get shifted on left hand side) and if k is negative signal is advanced in time k samples (Arrow get shifted on right hand side)

X(n) = { 1, -1 , 0 , 4 , -2 , 4 , 0 ,……}
↑
n=0

Delayed by 2 samples : X(n-2)= { 1, -1 , 0 , 4 , -2 , 4 , 0 ,……}
↑
n=0

Advanced by 2 samples : X(n+2) = { 1, -1 , 0 , 4 , -2 , 4 , 0 ,……}
↑
n=0

2) **Folding / Reflection** : It is folding of signal about time origin n=0. In this case replace n by –n.

Original signal: X(n) = { 1, -1 , 0 , 4 , -2 , 4 , 0}
↑
n=0

Folded signal: X(-n) = { 0 , 4 , -2 , 4 , 0 , -1 , 1}
↑
n=0

3) Addition : Given signals are x1(n) and x2(n), which produces output y(n) where y(n) = x1(n)+ x2(n). Adder generates the output sequence which is the sum of input sequences.

4) Scaling: Amplitude scaling can be done by multiplying signal with some constant. Suppose original signal is x(n). Then output signal is A x(n)

5) Multiplication: The product of two signals is defined as y(n) = x1(n) * x2(n).

Introduction to Z – Transform:

For analysis of continuous time LTI system Laplace transform is used. And for analysis of discrete time LTI system z transform is used. Z transform is mathematical tool used for conversion of time domain into frequency domain (z domain) and is a function of the complex valued variable Z. The z transform of a discrete time signal x(n) denoted by X(z) and given as

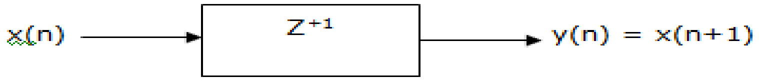
$$x(z)=\sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Z transform is an infinite power series because summation index varies from -∞ to ∞. But it is useful for values of z for which sum is finite. The values of z for which f (z) is finite and lie within the region called as “region of convergence (ROC).

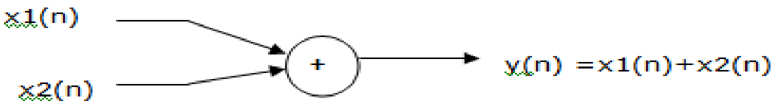
Symbols Used In Discrete Time System:

1. Unit delay

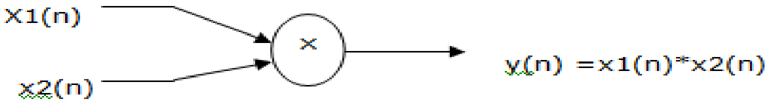
2. Unit advance



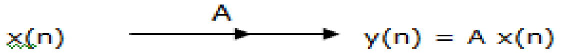
3. Addition



4. Multiplication



5. Scaling (constant multiplier)



Advantages of Z Transform:

1. The DFT can be determined by evaluating z transform.
2. Z transform is widely used for analysis and synthesis of digital filter.
3. Z transform is used for linear filtering. z transform is also used for finding Linear convolution, cross-correlation and auto-correlations of sequences.
4. In z transform user can characterize LTI system (stable/unstable, causal/anti- causal) and its response to various signals by placements of pole and zero plot.

Advantages of ROC(Region Of Convergence):

1. ROC is going to decide whether system is stable or unstable.
2. ROC decides the type of sequences causal or anti-causal.
3. ROC also decides finite or infinite duration sequences.

Z Transform Plot:

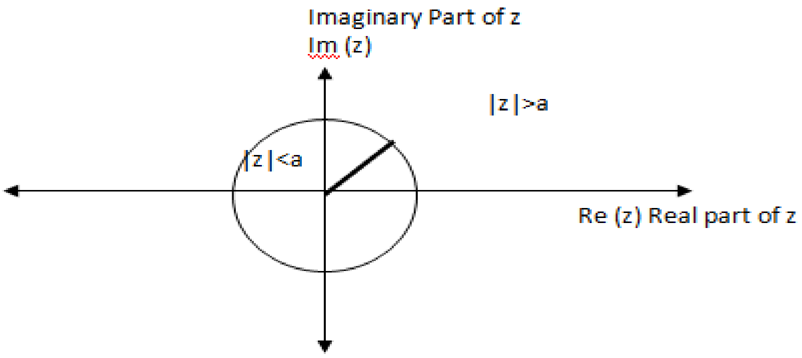


Fig show the plot of z transforms. The z transform has real and imaginary parts. Thus a plot of imaginary part versus real part is called complex z-plane. The radius of circle is 1 called as unit circle. This complex z plane is used to show ROC, poles and zeros. Complex variable z is also expressed in polar form as $Z= re^{j\omega}$ where r is radius of circle is given by |z| and ω is the frequency of the sequence in radians and given by ∟z.

S. No	Time Domain Sequence	Property	z Transform	ROC
1	δ(n) (Unit sample)		1	complete z plane
2	δ(n-k)	Time shifting	z^{-k}	except z=0
3	δ(n+k)	Time shifting	z^{k}	except z=∞
4	u(n) (Unit step)		$1/1- z^{-1} = z/z-1$	|z| > 1
5	u(-n)	Time reversal	1/1- z	|z| < 1
6	-u(-n-1)	Time reversal	z/z- 1	|z| < 1
7	n u(n) (Unit ramp)	Differentiation	$z^{-1} / (1- z^{-1})^2$	|z| > 1
8	a^n u(n)	Scaling	$1/1- (az^{-1})$	|z| > |a|
9	$-a^n$ u(-n-1)(Left side exponential sequence)		$1/1- (az^{-1})$	|z| < |a|
10	n a^n u(n)	Differentiation	$a z^{-1} / (1- az^{-}1)^2$	|z| > |a|
11	-n a^n u(-n-1)	Differentiation	$a z^{-1} / (1- az^{-}1)^2$	|z| < |a|
12	a^n for 0 < n < N-1		$1- (a z^{-1})^N/ 1- az^{-}1$	$\|az^{-}1\| < \infty$ except z=0
13	1 for 0<n<N-1 or u(n) – u(n-N)	Linearity Shifting	$1- z^{-N}/ 1- z^{-}1$	|z| > 1
14	$\cos(\omega_0 n)$ u(n)		$\frac{1- z^{-1}\cos\omega_0}{1- 2z^{-1}\cos\omega_0+z^{-2}}$	|z| > 1
15	$\sin(\omega_0 n)$ u(n)		$\frac{z^{-1}\sin\omega_0}{1- 2z^{-1}\cos\omega_0+z^{-2}}$	|z| > 1
16	$a^n \cos(\omega_0 n)$ u(n)	Time scaling	$\frac{1- (z/a)^{-1}\cos\omega_0}{1- 2(z/a)^{-1}\cos\omega_0+(z/a)^{-2}}$	|z| > |a|
17	$a^n \sin(\omega_0 n)$ u(n)	Time scaling	$\frac{(z/a)^{-1}\sin\omega_0}{1- 2(z/a)^{-1}\cos\omega_0+(z/a)^{-2}}$	|z| > |a|

Fourier Transform (FT):

Any signal can be decomposed in terms of sinusoidal (or complex exponential) components. Thus the analysis of signals can be done by transforming time domain signals into frequency domain and vice-versa. This transformation between time and frequency domain is performed with the help of Fourier Transform(FT) But still it is not convenient for computation by DSP processors hence Discrete Fourier Transform(DFT) is used.

Time domain analysis provides some information like amplitude at sampling instant but does not convey frequency content & power, energy spectrum hence frequency domain analysis is used.

For Discrete time signals x(n) , Fourier Transform is denoted as x(ω) & given by

$$x(\omega)= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

DFT is denoted by x(k) and given by (ω= 2 ∏ k/N)

$$X(k)= \sum_{n=0}^{N-1} x(n).e^{\frac{-j2\pi kn}{N}}$$

IDFT is given as

$$x(n)=\frac{1}{N}\sum_{k=0}^{N-1} X(k).e^{\frac{j2\pi kn}{N}}$$

3.2 DIFFERENCE BETWEEN FT & DFT

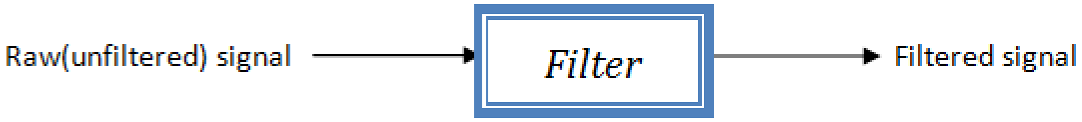
S.No	Fourier Transform (FT)	Discrete Fourier Transform (DFT)
1	FT x(ω) is the continuous function of x(n).	DFT x(k) is calculated only at discrete values of ω. Thus DFT is discrete in nature.
2	The range of ω is from - ∏ to ∏ or 0 to 2∏.	Sampling is done at N equally spaced points over period 0 to 2∏. Thus DFT is sampled version of FT.
3	FT is given by equation (1)	DFT is given by equation (2)
4	FT equations are applicable to most of infinite sequences.	DFT equations are applicable to causal, finite duration sequences
5	In DSP processors & computers applications of FT are limited because x(ω) is continuous function of ω.	In DSP processors and computers DFT's are mostly used. APPLICATION a) Spectrum Analysis b) Filter Design

Analog and digital filters

To remove or to reduce strength of unwanted signal like noise and to improve the quality of required signal filtering process is used. To use the channel full bandwidth we mix up two or more signals on transmission side and on receiver side we would like to separate it out in efficient way. Hence filters are used. Thus the digital filters are mostly used in

1. Removal of undesirable noise from the desired signals
2. Equalization of communication channels
3. Signal detection in radar, sonar and communication
4. Performing spectral analysis of signals.

The following block diagram illustrates the basic idea.



There are two main kinds of filter, analog and digital. They are quite different in their physical makeup and in how they work.

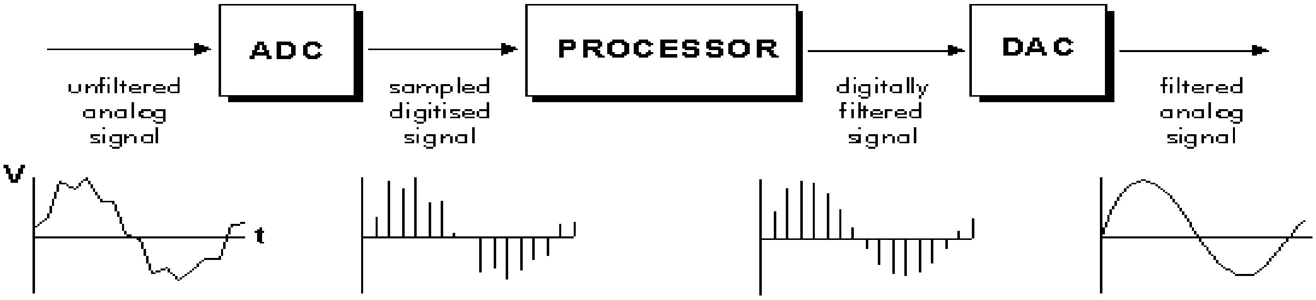
An analog filter uses analog electronic circuits made up from components such as resistors, capacitors and op amps to produce the required filtering effect. Such filter circuits are widely used in such applications as noise reduction, video signal enhancement, graphic equalizers in hi-fi systems, and many other areas.

In analog filters the signal being filtered is an electrical voltage or current which is the direct analogue of the physical quantity (e.g. a sound or video signal or transducer output) involved.

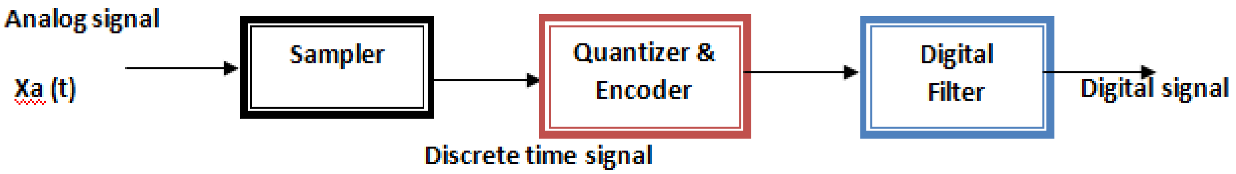
A digital filter uses a digital processor to perform numerical calculations on sampled values of the signal. The processor may be a general-purpose computer such as a PC, or a specialized DSP (Digital Signal Processor) chip.

The analog input signal must first be sampled and digitized using an ADC (analog to digital converter). The resulting binary numbers, representing successive sampled values of the input signal, are transferred to the processor, which carries out numerical calculations on them. These calculations typically involve multiplying the input values by constants and adding the products together. If necessary, the results of these calculations, which now represent sampled values of the filtered signal, are output through a DAC (digital to analog converter) to convert the signal back to analog form.

In a digital filter, the signal is represented by a sequence of numbers, rather than a voltage or current. The following diagram shows the basic setup of such a system.



BASIC BLOCK DIAGRAM OF DIGITAL FILTERS



1. Samplers are used for converting continuous time signal into a discrete time signal by taking samples of the continuous time signal at discrete time instants.
2. The Quantizer are used for converting a discrete time continuous amplitude signal into a digital signal by expressing each sample value as a finite number of digits.
3. In the encoding operation, the quantization sample value is converted to the binary equivalent of that quantization level.
4. The digital filters are the discrete time systems used for filtering of sequences. These digital filters performs the frequency related operations such as low pass,high pass, band pass and band reject etc. These digital Filters are designed with digital hardware and software and are represented by difference equation.

DIFFERENCE BETWEEN ANALOG FILTER AND DIGITAL FILTER

Sr No	Analog Filter	Digital Filter
1	Analog filters are used for filtering analog signals.	Digital filters are used for filtering digital sequences.
2	Analog filters are designed with various components like resistor, inductor and capacitor	Digital Filters are designed with digital hardware like FF, counters shift registers, ALU and software's like C or assembly language.
3	Analog filters less accurate & because of component tolerance of active components & more sensitive to environmental changes.	Digital filters are less sensitive to the environmental changes, noise and disturbances. Thus periodic calibration can be avoided. Also they are extremely stable.
4	Less flexible	These are most flexible as software programs & control programs can be easily modified. Several input signals can be filtered by one digital filter.
5	Filter representation is in terms of system components.	Digital filters are represented by the difference equation.
6	An analog filter can only be changed by redesigning the filter circuit.	A digital filter is programmable, i.e. its operation is determined by a program stored in the processor's memory. This means the digital filter can easily be changed without affecting the circuitry (hardware).

Multirate signal processing

The process of converting a signal from a given rate to a different rate is called sampling rate conversion. Systems that employ multiple sampling rates in the processing of digital signals are called multi rate digital signal processing.

Down sampling:

The process of reducing the sampling rate by an integer factor(D) is called decimation of the sampling rate. It is also called down sampling by factor(D).

Up sampling:

The process of increasing the sampling rate by an integer factor(I) is called interpolation of the sampling rate. It is also called up sampling by factor (I).

Table of Laplace and Z-transforms

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
1.	–	–	Kronecker delta $\delta_0(k)$ 1 $k = 0$ 0 $k \neq 0$	1
2.	–	–	$\delta_0(n\text{-}k)$ 1 $n = k$ 0 $n \neq k$	z^{-k}
3.	$\frac{1}{s}$	$1(t)$	$1(k)$	$\frac{1}{1-z^{-1}}$
4.	$\frac{1}{s+a}$	e^{-at}	e^{-akT}	$\frac{1}{1-e^{-aT}z^{-1}}$
5.	$\frac{1}{s^2}$	t	kT	$\frac{Tz^{-1}}{\left(1-z^{-1}\right)^2}$
6.	$\frac{2}{s^3}$	t^2	$(kT)^2$	$\frac{T^2z^{-1}\left(1+z^{-1}\right)}{\left(1-z^{-1}\right)^3}$
7.	$\frac{6}{s^4}$	t^3	$(kT)^3$	$\frac{T^3z^{-1}\left(1+4z^{-1}+z^{-2}\right)}{\left(1-z^{-1}\right)^4}$
8.	$\frac{a}{s(s+a)}$	$1 - e^{-at}$	$1 - e^{-akT}$	$\frac{\left(1-e^{-aT}\right)z^{-1}}{\left(1-z^{-1}\right)\left(1-e^{-aT}z^{-1}\right)}$
9.	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at} - e^{-bt}$	$e^{-akT} - e^{-bkT}$	$\frac{\left(e^{-aT}-e^{-bT}\right)z^{-1}}{\left(1-e^{-aT}z^{-1}\right)\left(1-e^{-bT}z^{-1}\right)}$
10.	$\frac{1}{(s+a)^2}$	te^{-at}	kTe^{-akT}	$\frac{Te^{-aT}z^{-1}}{\left(1-e^{-aT}z^{-1}\right)^2}$
11.	$\frac{s}{(s+a)^2}$	$(1 - at)e^{-at}$	$(1 - akT)e^{-akT}$	$\frac{1-\left(1+aT\right)e^{-aT}z^{-1}}{\left(1-e^{-aT}z^{-1}\right)^2}$
12.	$\frac{2}{(s+a)^3}$	t^2e^{-at}	$(kT)^2e^{-akT}$	$\frac{T^2e^{-aT}\left(1+e^{-aT}z^{-1}\right)z^{-1}}{\left(1-e^{-aT}z^{-1}\right)^3}$
13.	$\frac{a^2}{s^2(s+a)}$	$at - 1 + e^{-at}$	$akT - 1 + e^{-akT}$	$\frac{\left[\left(aT-1+e^{-aT}\right)+\left(1-e^{-aT}-aTe^{-aT}\right)z^{-1}\right]z^{-1}}{\left(1-z^{-1}\right)^2\left(1-e^{-aT}z^{-1}\right)}$
14.	$\frac{\omega}{s^2+\omega^2}$	$\sin \omega t$	$\sin \omega kT$	$\frac{z^{-1}\sin \omega T}{1-2z^{-1}\cos \omega T+z^{-2}}$
15.	$\frac{s}{s^2+\omega^2}$	$\cos \omega t$	$\cos \omega kT$	$\frac{1-z^{-1}\cos \omega T}{1-2z^{-1}\cos \omega T+z^{-2}}$
16.	$\frac{\omega}{(s+a)^2+\omega^2}$	$\mathrm{e}^{-at}\sin \omega t$	$\mathrm{e}^{-akT}\sin \omega kT$	$\frac{e^{-aT}z^{-1}\sin \omega T}{1-2e^{-aT}z^{-1}\cos \omega T+e^{-2aT}z^{-2}}$
17.	$\frac{s+a}{(s+a)^2+\omega^2}$	$\mathrm{e}^{-at}\cos \omega t$	$\mathrm{e}^{-akT}\cos \omega kT$	$\frac{1-e^{-aT}z^{-1}\cos \omega T}{1-2e^{-aT}z^{-1}\cos \omega T+e^{-2aT}z^{-2}}$
18.	–	–	a^k	$\frac{1}{1-az^{-1}}$
19.	–	–	a^k $k = 1, 2, 3, \dots$	$\frac{z^{-1}}{1-az^{-1}}$
20.	–	–	ka^{k-1}	$\frac{z^{-1}}{\left(1-az^{-1}\right)^2}$
21.	–	–	k^2a^{k-1}	$\frac{z^{-1}\left(1+az^{-1}\right)}{\left(1-az^{-1}\right)^3}$
22.	–	–	k^3a^{k-1}	$\frac{z^{-1}\left(1+4az^{-1}+a^2z^{-2}\right)}{\left(1-az^{-1}\right)^4}$
23.	–	–	k^4a^{k-1}	$\frac{z^{-1}\left(1+11az^{-1}+11a^2z^{-2}+a^3z^{-3}\right)}{\left(1-az^{-1}\right)^5}$
24.	–	–	$a^k \cos k\pi$	$\frac{1}{1+az^{-1}}$

$x(t) = 0$ for $t < 0$
$x(kT) = x(k) = 0$ for $k < 0$
Unless otherwise noted, $k = 0, 1, 2, 3, \dots$

Definition of the Z-transform

$$\mathscr{Z}\{x(k)\} = X(z) = \sum_{k=0}^{\infty} x(k)z^{-k}$$

Important properties and theorems of the Z-transform

	$x(t)$ or $x(k)$	**$\mathcal{Z}\{x(t)\}$ or $\mathcal{Z}\{x(k)\}$**
1.	$ax(t)$	$aX(z)$
2.	$ax_1(t)+bx_2(t)$	$aX_1(z)+bX_2(z)$
3.	$x(t+T)$ or $x(k+1)$	$zX(z)-zx(0)$
4.	$x(t+2T)$	$z^2X(z)-z^2x(0)-zx(T)$
5.	$x(k+2)$	$z^2X(z)-z^2x(0)-zx(1)$
6.	$x(t+kT)$	$z^kX(z)-z^kx(0)-z^{k-1}x(T)-\ldots-zx(kT-T)$
7.	$x(t-kT)$	$z^{-k}X(z)$
8.	$x(n+k)$	$z^kX(z)-z^kx(0)-z^{k-1}x(1)-\ldots-zx(k1-1)$
9.	$x(n-k)$	$z^{-k}X(z)$
10.	$tx(t)$	$-Tz\frac{d}{dz}X(z)$
11.	$kx(k)$	$-z\frac{d}{dz}X(z)$
12.	$e^{-at}x(t)$	$X(ze^{aT})$
13.	$e^{-ak}x(k)$	$X(ze^{a})$
14.	$a^kx(k)$	$X\left(\frac{z}{a}\right)$
15.	$ka^kx(k)$	$-z\frac{d}{dz}X\left(\frac{z}{a}\right)$
16.	$x(0)$	$\lim_{z\to\infty} X(z)$ if the limit exists
17.	$x(\infty)$	$\lim_{z\to 1}\left[\left(1-z^{-1}\right)X(z)\right]$ if $\left(1-z^{-1}\right)X(z)$ is analytic on and outside the unit circle
18.	$\nabla x(k)=x(k)-x(k-1)$	$\left(1-z^{-1}\right)X(z)$
19.	$\Delta x(k)=x(k+1)-x(k)$	$(z-1)X(z)-zx(0)$
20.	$\sum_{k=0}^{n} x(k)$	$\frac{1}{1-z^{-1}}X(z)$
21.	$\frac{\partial}{\partial a}x(t,a)$	$\frac{\partial}{\partial a}X(z,a)$
22.	$k^mx(k)$	$\left(-z\frac{d}{dz}\right)^m X(z)$
23.	$\sum_{k=0}^{n} x(kT)y(nT-kT)$	$X(z)Y(z)$
24.	$\sum_{k=0}^{\infty} x(k)$	$X(1)$

Table of Laplace Transforms

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n, \quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p, p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}}, \quad n=1,2,3,\ldots$	$\frac{1\cdot3\cdot5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{\left(s^2+a^2\right)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{\left(s^2+a^2\right)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{\left(s^2+a^2\right)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{\left(s^2+a^2\right)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s\left(s^2-a^2\right)}{\left(s^2+a^2\right)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s\left(s^2+3a^2\right)}{\left(s^2+a^2\right)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\frac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\frac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22.	$\mathbf{e}^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at}, \quad n=1,2,3,\ldots$	$\frac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\frac{\mathbf{e}^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	$\mathbf{e}^{-cs}$
27.	$u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28.	$u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathfrak{L}\{g(t+c)\}$
29.	$\mathbf{e}^{ct}f(t)$	$F(s-c)$	30.	$t^n f(t), \quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31.	$\frac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\frac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\frac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			

Table Notes

1. This list is not a complete listing of Laplace transforms and only contains some of the more commonly used Laplace transforms and formulas.

2. Recall the definition of hyperbolic functions.

$$\cosh(t) = \frac{\mathbf{e}^{t} + \mathbf{e}^{-t}}{2} \qquad\qquad \sinh(t) = \frac{\mathbf{e}^{t} - \mathbf{e}^{-t}}{2}$$

3. Be careful when using "normal" trig function vs. hyperbolic functions. The only difference in the formulas is the "+ a^2" for the "normal" trig functions becomes a "- a^2" for the hyperbolic functions!

4. Formula #4 uses the Gamma function which is defined as

$$\Gamma(t) = \int_0^\infty \mathbf{e}^{-x} x^{t-1}\, dx$$

   If n is a positive integer then,

$$\Gamma(n+1) = n!$$

   The Gamma function is an extension of the normal factorial function. Here are a couple of quick facts for the Gamma function

$$\Gamma(p+1) = p\Gamma(p)$$

$$p(p+1)(p+2)\cdots(p+n-1) = \frac{\Gamma(p+n)}{\Gamma(p)}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$