GPCET

Pioneering Innovative Education

G. PULLAIAH COLLEGE OF ENGINEERING AND TECHNOLOGY

Accredited by NAAC with 'A' Grade of UGC, Approved by AICTE, New Delhi

Permanently Affiliated to JNTUA, Ananthapuramu

(Recognized by UGC under 2(f) and 12(B) & ISO 9001:2008 Certified Institution)

Nandikotkur Road, Venkayapalli, Kurnool – 518452

Department of Humanities & Sciences

Bridge Course
On
MATHEMATICS

GPCET
Pioneering Innovative Education

1. MATRICES

INTRODUCTION OF MATRICES:

A simple form of matrices may have been used by the Mayans (and maybe other cultures; see below), the true mathematical use of a matrix was first formulated around 1850, by English mathematician, poet, and **musician James Sylvester** (1814–1897).

APPLICATIONS OF MATRICES IN REAL WORLD: In everyday **applications**, **matrices are used** to represent **real**-world data, such as the traits and habits of a certain population. They are **used** in geology to measure seismic waves. **Matrices** are rectangular arrangements of expressions, numbers and symbols that are arranged in columns and rows.

DEFINATION: A *matrix* is a rectangular array of elements. The elements can be symbolic expressions or/and numbers. Matrix $[A]$ is denoted by

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$ is called an m×n *matrix*.

e.g. $\begin{bmatrix} 2 & 3 & 4 \\ 1 & -8 & 5 \end{bmatrix}$ is a 2×3 matrix.

e.g. $\begin{bmatrix} 2 \\ 7 \\ -3 \end{bmatrix}$ is a 3×1 matrix.

Order of a Matrix: - A matrix having m rows and n columns is called a matrix of order m× n or simply m× n matrix (read as an **m by n matrix).**

e.g. $\begin{bmatrix} 2 & 0 & 3 & 6 \\ 3 & 4 & 7 & 0 \\ 1 & 9 & 2 & 5 \end{bmatrix}$ is a matrix of order 3×4.

e.g. $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 5 \\ -1 & 3 & 0 \end{bmatrix}$ is a matrix of order 3.

Special Types of matrices:

1. Square matrix: A matrix in which numbers of rows are equal to number of columns is called a square matrix.

$$Ex: \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad B = \begin{pmatrix} 1 & 3 & 5 \\ -6 & 8 & 9 \\ 5 & 1 & -6 \end{pmatrix}$$

2. Diagonal matrix: A square matrix A = $(a_{ij})_{n\,X\,n}$ is called a diagonal matrix if each of its non-diagonal element is zero. That is $a_{ij} = 0$, if $i \neq j$ and at least one element $a_{ii} \neq 0$

$$Ex: \quad A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

3. Identity Matrix : A diagonal matrix whose diagonal elements are equal to 1 is called identity matrix and denoted by I_n . that is $a_{ij} = \begin{cases} 0, i \neq j \\ 1, i = j \end{cases}$

$$Ex: \; I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Upper Triangular matrix :

A square matrix said to be a Upper triangular matrix if $a_{ij} = 0 \; if \;\; i > j$

$$Ex: \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 8 & 2 \\ 0 & 0 & -6 \end{pmatrix}$$

5. Lower Triangular Matrix:

A square matrix said to be a Lower triangular matrix if $a_{ij} = 0 \; if \;\; i < j$

$$Ex: \quad A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ -6 & 8 & 0 \\ 5 & 1 & -6 \end{pmatrix}$$

6. Symmetric Matrix: A square matrix $A = (a_{ij})_{m\,X\,n}$ said to be a symmetric if $a_{ij} = a_{ji} \;\; for\; all\; i\; and\; j$

$$Ex: \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \qquad B = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 8 & 9 \\ 5 & 9 & -6 \end{pmatrix}$$

7. Skew-Symmetric Matrix: A square matrix $A = (a_{ij})_{m\,X\,n}$ said to be a symmetric if $a_{ij} = -a_{ji} \;\; for\; all\; i\; and\; j$

$Ex: \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ -a_{12} & a_{22} & a_{23} \\ -a_{13} & -a_{23} & a_{33} \end{bmatrix}$ $\qquad B = \begin{pmatrix} 1 & 3 & 5 \\ -3 & 8 & 9 \\ -5 & -9 & -6 \end{pmatrix}$

8. Zero Matrix: A matrix whose all elements are zero is called as Zero Matrix and order $n\ X\ m$ Zero matrix denoted by $O_{n\ X\ m}$.

$Ex: O_{3\ X\ 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

9. Row Matrix: A matrix consists a single row is called as a row vector or row matrix.

$Ex: \quad A = [a_{11} \quad a_{12} \quad a_{13}] \qquad B = (1 \quad 2 \quad 3)$

10. Column Matrix: A matrix consists a single column is called a column vector or column matrix

$Ex: \ A = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} \qquad B = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

11. Submatrix: If some row(s) or/and column(s) of a matrix $[A]$ are deleted (no rows or columns may be deleted), the remaining matrix is called a submatrix of $[A]$.

$\boldsymbol{Ex}: \mathbf{1} \quad [A] = \begin{bmatrix} 4 & 6 & 2 \\ 3 & -1 & 2 \end{bmatrix}$

Solution:-

$\begin{bmatrix} 4 & 6 & 2 \\ 3 & -1 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 6 \\ 3 & -1 \end{bmatrix}, [4 \quad 6 \quad 2], [4], \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ are some of the sub matrices of $[A]$

12. *Powers of matrices*: For any square matrix A and any positive integer n, the symbol A^n denotes $\underbrace{A \cdot A \cdot A \cdots A}_{n \text{ factors}}$.

Note: (1) $\quad (A+B)^2 = (A+B)(A+B)$

$$= AA + AB + BA + BB$$
$$= A^2 + AB + BA + B^2$$

(2) If $AB = BA$, then $(A+B)^2 = A^2 + 2AB + B^2$

13. DETERMINANTS: Let $A = \left[a_{ij}\right]$ be a square matrix of order n. The determinant of A, detA or |A| is defined as follows:

(a) If n=2, $\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

(b) If n=3, $\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

or $$= -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

14. Multiplication of Determinants: Let $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, $|B| = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$ Then

$$|A||B| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = \begin{vmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{vmatrix}$$

Properties :

(1) det(AB)=(detA)(detB) i.e. $|AB| = |A||B|$

(2) |A|(|B||C|)=(|A||B|)|C| N.B. A (BC)=(AB)C

(3) |A||B|=|B||A| N.B. AB≠BA in general

(4) |A|(|B|+|C|)=|A||B|+|A||C| N.B. A (B+C)=AB+AC

15. <u>INVERSE MATRIX</u>:

i) The inverse of a 2×2 matrix A, is another 2×2 matrix denoted by A^{-I} with the property that $A^{-I}A = AA^{-I} = I$. The inverse of a 2×2 matrix can also be determined.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ represents any 2×2 matrix, then the inverse of **A**, written as **A**$^{-1}$, is found by

$$A^{-1} = \begin{pmatrix} \dfrac{d}{\det A} & \dfrac{-b}{\det A} \\ \dfrac{-c}{\det A} & \dfrac{a}{\det A} \end{pmatrix}$$

When a 2×2 matrix is multiplied by its inverse, the result is the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

ii) Let A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ The Inverse of a 3 X 3 Matrix is $A^{-I} = \frac{1}{|A|} Adj\ A$

Minors and cofactors of a Matrix

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Minor of $a_{ij} \equiv M_{ij}$, is determinant obtained by deleting ith row and jth column.

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \text{ is determinant obtained by deleting 1st row and 1st column}$$

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Signs of Cofactors

- For 2x2 – matrix $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$
- For 3x3 – matrix $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$
- For 4x4 – matrix $\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$

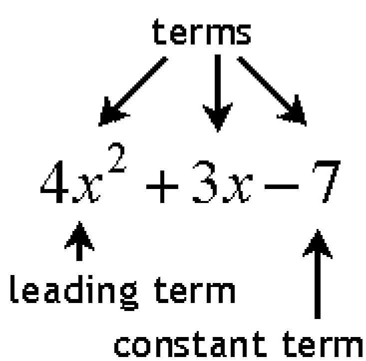
2. POLYNOMIALS

Polynomials are one of the most important concepts in algebra and throughout mathematics and science. They are used to form polynomial equations, which encode a wide range of problems, from elementary word problems to complicated problems in the sciences; they are used to define polynomial functions, which appear in settings ranging from basic chemistry and physics to economics, and are used in calculus and numerical analysis to approximate other functions. Polynomials are used to construct polynomial rings, one of the most powerful concepts in algebra and algebraic geometry.

By now, you should be familiar with variables and exponents, and you may have dealt with expressions like $3x^4$ or $6x$. Polynomials are sums of these "variables and exponents" expressions. Each piece of the polynomial, each part that is being added, is called a "term". Polynomial terms have variables which are raised to whole-number exponents (or else the terms are just plain numbers); there are no square roots of variables, no fractional powers, and no variables in the denominator of any fractions. Here are some examples:

$6x^{-2}$	This is NOT a polynomial term...	...because the variable has a negative exponent.
$^1/_{x^2}$	This is NOT a polynomial term...	...because the variable is in the denominator.
sqrt(x)	This is NOT a polynomial term...	...because the variable is inside a radical.
$4x^2$	This IS a polynomial term...	...because it obeys all the rules.

Here is a typical polynomial:



- Give the degree of the following polynomial: $2x^5 - 5x^3 - 10x + 9$

  This polynomial has four terms, including a fifth-degree term, a third-degree term, a first-degree term, and a constant term.

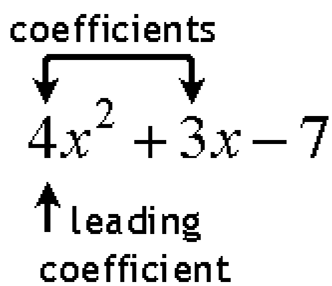
  This is a fifth-degree polynomial.

- Give the degree of the following polynomial: $7x^4 + 6x^2 + x$

  This polynomial has three terms, including a fourth-degree term, a second-degree term, and a first-degree term. There is no constant term.

  This is a fourth-degree polynomial.

When a term contains both a number and a variable part, the number part is called the "coefficient". The coefficient on the leading term is called the "leading" coefficient.



In the above example, the coefficient of the leading term is 4; the coefficient of the second term is 3; the constant term doesn't have a coefficient.

The "poly" in "polynomial" means "many". I suppose, technically, the term "polynomial" should only refer to sums of *many* terms, but the term is used to refer to anything from one term to the sum of a zillion terms. However, the shorter polynomials do have their own names:

- a one-term polynomial, such as $2x$ or $4x^2$, may also be called a "monomial" ("mono" meaning "one")
- a two-term polynomial, such as $2x + y$ or $x^2 - 4$, may also be called a "binomial" ("bi" meaning "two")
- a three-term polynomial, such as $2x + y + z$ or $x^4 + 4x^2 - 4$, may also be called a "trinomial" ("tri" meaning "three")

Polynomials classified by degree

Degree	Name	Example
$-\infty$	zero	0
0	(non-zero) constant	1
1	linear	$x + 1$
2	quadratic	$x^2 + 1$
3	cubic	$x^3 + 1$
4	quartic (or biquadratic)	$x^4 + 1$
5	quintic	$x^5 + 1$
6	sextic or hexic	$x^6 + 1$
7	septic or heptic	$x^7 + 1$
8	octic	$x^8 + 1$
9	nonic	$x^9 + 1$
10	decic	$x^{10} + 1$

Usually, a polynomial of degree 4 or higher is referred to as a *polynomial of degree n*, although the phrases *quartic polynomial* and *quintic polynomial* are also used. The names for degrees higher than 5 are even less common. The names for the degrees may be applied to the polynomial or to its terms. For example, a constant may refer to a zero degree polynomial or to a zero degree term.

The polynomial 0, which may be considered to have no terms at all, is called the **zero polynomial**. Unlike other constant polynomials, its degree is not zero. Rather the degree of the zero polynomial is either left explicitly undefined, or defined to be negative (either –1 or –∞).The latter convention is important when defining Euclidean division of polynomials.

Polynomials classified by number of non-zero terms

Number of non-zero terms	Name	Example
0	zero polynomial	0
1	monomial	x^2
2	binomial	$x^2 + 1$
3	trinomial	$x^2 + x + 1$

Further, polynomials may be classified by the number of terms (using the minimal number of terms, that is, not counting zero terms and combining like terms). The word *monomial* can be ambiguous, used either to refer to a polynomial with just a single term, as above, or to refer to the particular case of monic monomials, that is, having coefficient 1.

GPCET
Pioneering Innovative Education

3. Binomial Theorem

The Binomial Theorem is used to expand out brackets of the form $(a+b)^n$, where *n* is a whole number.

n	$(a+b)^n$	*Coefficients*
0	$(a+b)^0 = 1$	1
1	$(a+b)^1 = a+b$	1 1
2	$(a+b)^2 = a^2+2ab+b^2$	1 2 1
3	$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$	1 3 3 1
4	$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$	1 4 6 4 1

Note 1: The coefficients in these expansions form **Pascal's Triangle**. These numbers can also be found using the $\boxed{{}^nC_r}$ button on a calculator. For example, the coefficients for the expansion of $(a+b)^7$ are:

$${}^7C_0 = 1 \quad {}^7C_1 = 7 \quad {}^7C_2 = 21 \quad {}^7C_3 = 35$$

$${}^7C_4 = 35 \quad {}^7C_5 = 21 \quad {}^7C_6 = 7 \quad {}^7C_7 = 1$$

Note 2: As the power of *a* decreases by 1, the power of *b* increases by 1. In each term, when you add together the powers of *a* and *b* together you get *n*.

So, $(a+b)^7 = a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7$

The binomial expansion is the series expansion of $(a+b)^n$. There are different versions depending on whether n is an integer or a real number.

Binomial expansion for integer *n*

This is a *finite* series given by:

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$$

$$= b^n + nab^{n-1} + \frac{n(n-1)}{2!}a^2b^{n-2} + \ldots + na^{n-1}b + a^n$$

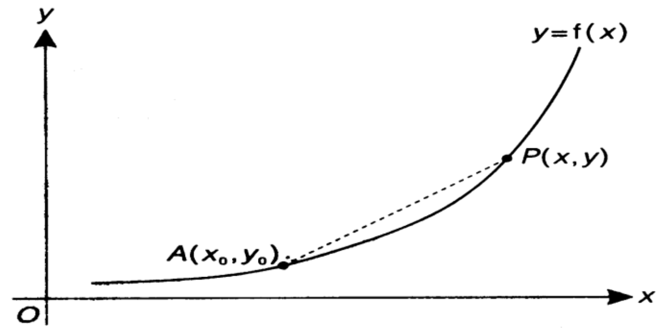
Binomial expansion for real *n*

This is an *infinite* series expansion of $(1+x)^n$, valid *only* when $|x| < 1$:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

4. DIFFERENTIATION

INTRODUCTION



Let $A(x_0, y_0)$ be a fixed point and $P(x, y)$ be a variable point on the curve $y = f(x)$ as shown on about figure. Then the slope of the line *AP* is given by $\frac{y - y_0}{x - x_0}$ or $\frac{f(x) - f(x_0)}{x - x_0}$. When the variable point *P* moves closer and closer to *A* along the curve $y = f(x)$, i.e. $x \to x_0$. the line *AP* becomes the tangent line of the curve at the point *A*. Hence, the slope of the tangent line at the point *A* is equal to $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$. This term is defined to be the derivative of $f(x)$ at $x = x_0$ and is usually denoted by $f'(x_0)$. The definition of derivative at any point x may be defined as follows.

Definition Let $y = f(x)$ be a function defined on the interval $[a,b]$ and $x_0 \in (a,b)$.

$f(x)$ is said to be differentiable at x_0 (or have a derivative at x_0) if the limit $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists. This lime value is denoted by $f'(x_0)$ or $\left.\frac{dy}{dx}\right|_{x=x_0}$ and is called the derivative of $f(x)$ at x_0.

If $f(x)$ has a derivative at every point x in (a,b), then $f(x)$ is said to be differentiable on (a,b).

Remark As $x \to x_0$, the difference between x and x_0 is very small, i.e. $x - x_0$ tends to zero. Usually, this difference is denoted by h or Δx. Then the derivative at x_0 may be rewritten as $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$. (First Principle)

RULES OF DIFFERENTIATION

Composite functions

$$\frac{d}{dx}ku = k\frac{du}{dx} \qquad \frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx} \qquad \frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \qquad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

Algebraic functions

$$\frac{d}{dx}x^k = kx^{k-1}$$

where ***k*** must be ***independent*** of ***x*** (usually a constant)

Inverse functions *(esp.: inverse of trigo func)*

If $y = f^{-1}(x)$ then $\dfrac{dy}{dx} = \dfrac{1}{\frac{df(y)}{dy}}$

Trigonometric functions

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\sec x = \sec x \tan x \qquad \frac{d}{dx}\csc x = -\csc x \cot x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

Inverse Trigonometric functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}, \qquad \frac{d}{dx}\cot^{-1} x = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}, \qquad \frac{d}{dx}\csc^{-1} x = \frac{-1}{|x|\sqrt{x^2-1}}$$

Logarithmic functions

$$\frac{d}{dx}e^x = e^x \qquad\qquad \frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{d}{dx}a^x = a^x \ln a \qquad\qquad \frac{d}{dx}\log_a x = \frac{1}{x \ln a}$$

Parametric functions (commonly use in Rate of change)

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Leibniz's Theorem Let f and g be two functions with nth derivative. Then

$$\frac{d^n}{dx^n}[f(x)g(x)] = \sum_{r=0}^{n} C_r^n f^{(r)}(x) g^{(n-r)}(x) \text{ where } f^{(0)}(x) = f(x).$$

MEAN VALUE THEOREM:

Let $y = f(x)$ be a function defined on an interval **I**. f is said to have an ***absolute maximum*** at *c* if $f(c) \geq f(x), \forall x \in$ **I** and $f(c)$ is called the ***absolute maximum value***. Similarly, *f* is said to have an ***absolute minimum*** at *d* if $f(d) \leq f(x), \forall x \in$ **I** and $f(d)$ is called the ***absolute minimum value***.

Theorem: - **Rolle's Theorem**

If a function $f(x)$ satisfies all the following three conditions:

(1) $f(x)$ is continuous on the closed interval $[a,b]$,

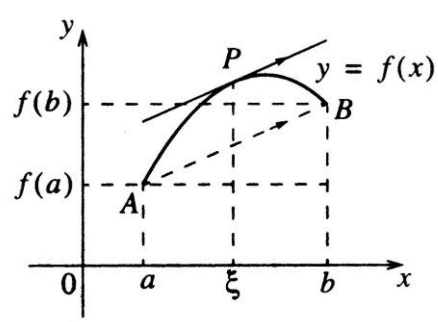
(2) $f(x)$ is differentiable in the open interval (a,b),

(3) $f(a) = f(b)$; then there exists at least a point $\xi \in (a,b)$ such that $f'(\xi) = 0$.

Theorem:-Mean Value Theorem

If a function $f(x)$ is (1) continuous on the closed interval $[a,b]$ and

(2) differentiable in the open interval (a,b),



then there exists at least a point $\xi \in (a,b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(\xi).$$

GPCET

Pioneering Innovative Education

5. INTEGRATION

I) Integrals of Rational and Irrational Functions.

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$
2. $\int c\, dx = cx + C$
3. $\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + C$
4. $\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + C$
5. $\int f^n(x)\, f'(x) dx = \frac{f^{n+1}(x)}{n+1} + C$
6. $\int \frac{1}{1+x^2} dx = \mathrm{Tan}^{-1} x + C$
7. $\int \frac{1}{\sqrt{1-x^2}} dx = \mathrm{Sin}^{-1} x + C$
8. $\int \frac{-1}{\sqrt{1-x^2}} dx = \mathrm{Cos}^{-1} x + C$
9. $\int u.v\, dx = u \int v dx - \int \left[\frac{d}{dx}(u) \int v dx\right] dx$

II) Integrals of Trigonometric Functions.

1. $\int \mathrm{Sin}\, x dx = -\mathrm{Cos}\, x + C$
2. $\int \mathrm{Cos}\, x dx = \mathrm{Sin}\, x + C$
3. $\int Tan\, x dx = \log|\sec x| + C$
4. $\int \mathrm{Sec}\, x dx = \log|\sec x + \tan x| + C$
5. $\int Cotx\, dx = \log|\mathrm{Sin}\, x| + C$
6. $\int Cosec\, x\, dx =$
7. $\int Tan^2\, x dx = Tanx - x + C$
8. $\int Sec^2\, x\, dx = Tanx + C$

III) Integrals of Exponential and Logarithmic Functions.

1. $\int \log x dx = x \log x - x + C$
2. $\int e^x dx = e^x + C$
3. $\int a^x dx = \frac{a^x}{\log a} + C$
4. $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\ \mathrm{Sin}\, bx - b\ Cosbx] + C$
5. $\int e^{ax} \mathrm{Cos}\, bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\ \mathrm{Cos}\, bx + b\ Sinbx] + C$

GPCET

Pioneering Innovative Education

iv) Standard Results :

1. $\int \frac{1}{x} dx = \ln x + c$

2. $\int a^x dx = \frac{a^x}{\ln a} + c$

3. $\int \sec x \tan x dx = \sec x + c$

4. $\int \csc^2 x dx = -\cot x + c$

5. $\int \csc x \cot x dx = -\csc x + c$

6. $\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left|\frac{x + \sqrt{x^2 - a^2}}{a}\right| + c$

7. $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} + c$

8. $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

9. $\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left|\frac{\sqrt{x^2 + a^2} + x}{a}\right| + c$

10. $\int u dv = uv - \int v du.$

GPCET

Pioneering Innovative Education

6. COMPLEX NUMBERS

(1) A ***complex number*** z is a number of the form $a+bi$ where a, b are real numbers and $i^2=-1$.

(2) The set **C** of all complex numbers is defined by **C**= $\{a+bi : a,b \in R \text{ and } i^2=-1\}$

where a is called the ***real part*** of z and $a=\mathrm{Re}(z)$ and

b is called the ***imaginary part*** of z and $b=\mathrm{Im}(z)$.

(3) z is said to be ***purely imaginary*** if and only if $\mathrm{Re}(z)=0$ and $\mathrm{Im}(z)\neq 0$.

(4) When $\mathrm{Im}(z)=0$, the complex number z is ***real***.

NOTE: $i^3=i^2\cdot i=-i$, $i^4=i^2\cdot i^2=1$, $i^5=i^4\cdot i=i$, $i^6=i^4\cdot i^2=-1$.

Operations On Complex Numbers

Let $z_1=a+bi$ and $z_2=c+di$. Then

(1) $z_1+z_2=(a+c)+(b+d)i$

(2) $z_1-z_2=(a-c)+(b-d)i$

(3) $z_1z_2 = (a+bi)(c+di)$

$= (ac-bd)+(ad+bc)i$

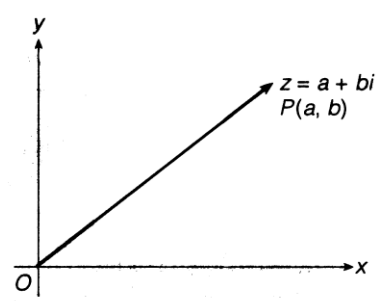
(4) $\dfrac{z_1}{z_2} = \dfrac{a+bi}{c+di}\cdot\dfrac{c-di}{c-di} = \dfrac{ac+bd}{c^2+d^2}+\dfrac{(bc-ad)}{c^2+d^2}i$, where $z_2\neq 0$.

NOTE: (i) $\dfrac{1}{i}=\dfrac{i}{i^2}=\dfrac{i}{-1}=-i$;

(ii) $\dfrac{1}{z_2}=\dfrac{1}{c+di}=\dfrac{c-di}{c^2+d^2}=\dfrac{1}{c^2+d^2}\overline{z_2}$.

Geometrical Representation of a Complex Number:

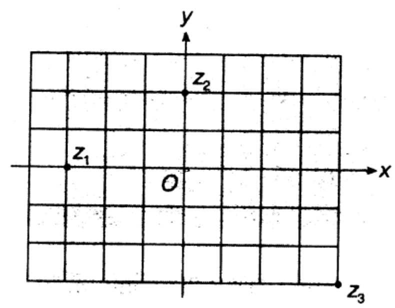
From the definition of complex numbers, a complex number $z=a+bi$ is defined by the two real numbers a and b. Hence, if we consider the real part a as the $x-coordinate$ in the rectangular coordinates system and the imaginary part b as the $y-coordinate$, then the complex number z can be represented by the point (a,b) on the plane. This plane is called the ***complex plane*** or the ***Argand diagram***. On this plane, real numbers are represented by points on $x-axis$ which is called the ***real axis***; imaginary numbers are represented by points on the $y-axis$ which is called the ***imaginary axis***. The number 0 is represented by the origin ***O***.



Any point (a,b) on this plane can be used to represent a complex number $z=a+bi$.

For example, as shown in Figure, the z_1, z_2, z_3 represents respectively the complex numbers.

$$z_1 = -3, z_2 = 2i, z_3 = 4 - 3i$$



Polar Form of a Complex Number

A. Polar Form

A complex number $z = a + bi$ can be represented by a vector $\overrightarrow{OP}$ as shown in Figure.

The length of the vector $\overrightarrow{OP}$, $r = \left|\overrightarrow{OP}\right|$, is called the modulus of the complex number z, and it is denoted by $|z|$. The angle between the vector $\overrightarrow{OP}$ and the positive real axis is defined to be the argument or amplitude of z and is denoted by $\arg z$ or amp z.

$\arg z$ is infinitely many-valued, that is, $\arg z = \theta + 2k\pi$, *where* $k \in Z$.

If $\arg z$ lies in the interval $-\pi < \theta \le \pi$, we call this value the ***principal value***.

Complex Conjugate:

Definition: Let $z = a + bi$, where $a, b \in R$. The complex conjugate of z, denoted by $\bar{z}$ is defined as $\bar{z} = a - bi$

Theorem: Properties of Complex Conjugate

Let z be a complex number. Then

(1) z is real if and only if $\bar{z} = z$.

(2) $\bar{\bar{z}} = z$

(3) $z\bar{z} = |z|^2$

(4) $|\bar{z}| = |z|$

(5) $\arg \bar{z} = -\arg z \ (z \neq 0)$

(6) $z + \bar{z} = 2\,\mathrm{Re}(z)$

(7) $z - \bar{z} = 2i\,\mathrm{Im}(z)$

GPCET
Pioneering Innovative Education

8. Integration and partial fractions

(I) The method

This method allows you to integrate functions of the form $\dfrac{1}{(x+2)(x-3)^2}$.

Note that neither substitution nor integration by parts is likely to help, here.

However, it is possible to split $\dfrac{1}{(x+2)(x-3)^2}$ into several fractions, which are easier to integrate.

It is easier to explain (and to understand) how it works through examples rather than to give an overview of the rigorous mathematical theory.

First, the degree of the numerator has to be smaller than the degree of the denominator. They are three cases:

(i) The denominator of the fraction contains only simple binomials. Example:

$$\frac{2x}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3}.$$

(ii) The denominator of the fraction contains a repeated binomial. Example:

$$\frac{1}{(x+2)(x-3)^2} = \frac{A}{x+2} + \frac{B}{x-3} + \frac{C}{(x-3)^2}.$$

(iii) The denominator of the fraction contains an irreducible quadratic. Example:

$$\frac{1}{(x+2)(x^2+x+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+x+1}.$$

By irreducible quadratic, we mean here a quadratic with no real root.
Once the fraction has been split into smaller fractions, it is just a matter of finding the coefficients.

<u>Example</u>:Let us consider $\dfrac{2x}{(x+2)(x-3)}$. According to the above,

$$\frac{2x}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3}.$$

Now, multiply both sides of the equality by $(x+2)(x-3)$:

$2x = A(x-3) + B(x+2),$

and, after rearrangements, $2x = x(A+B) + (2B-3A)$.

The polynomial on the right hand side is identically equal to the polynomial on the left hand side if and only if their coefficients are equal, i.e. if and only if

$\begin{cases} A+B=2 \\ 2B-3A=0 \end{cases}$ .Solving for A and B, we obtain $A=\dfrac{4}{5}$ and $B=\dfrac{6}{5}$ so that

$$\frac{2x}{(x+2)(x-3)} = \frac{4}{5(x+2)} + \frac{6}{(x-3)}.$$

Hence $\displaystyle\int \frac{2x}{(x+2)(x-3)}dx = \int \frac{4dx}{5(x+2)} + \int \frac{6dx}{5(x-3)} = \frac{4}{5}\ln(x+2) + \frac{6}{5}\ln(x-3).$

GPCET
Pioneering Innovative Education

9.TRIGONOMETRY IDENTITIES

1.$sin\theta = \frac{opposite}{hypotenuse}$

2.$cos\theta = \frac{adjacent}{hypotenuse}$

3.$tan\theta = \frac{opposite}{adjacent}$

4.$cosec\theta = \frac{hypotenuse}{opposite}$

5.$sec\theta = \frac{hypotenuse}{adjacent}$

6.$cot\theta = \frac{adjacent}{opposite}$

7.$\sin(-\theta) = -sin\theta$

8.$\cos(-\theta) = cos\theta$

9.$\sin(\pi - \theta) = sin\theta$

10.$cos(\pi - \theta) = -cos\theta$

11. $\sin(\pi + \theta) = -sin\theta$

12. $cos(\pi + \theta) = -cos\theta$

13.$sin^2\theta + cos^2\theta = 1$

14.$1 + tan^2\theta = sec^2\theta$

15.1+$cot^2\theta = cosec^2\theta$

16.$\sin(A + B) = sinAcosB + cosAsinB$

17.$\sin(A - B) = sinAcosB - cosAsinB$

18.$\cos(A + B) = cosAcosB - sinAsinB$

19. $\cos(A - B) = cosAcosB + sinAsinB$

20. $\tan(A+B) = \frac{tanA+tanB}{1-tanAtanB}$

21. $\tan(A-B) = \frac{tanA-tanB}{1+tanAtanB}$

22. $\sin(2\theta) = 2sin\theta cos\theta$

23. $\cos(2\theta) = (cos^2\theta - sin^2\theta)$ or $(1 - 2sin^2\theta)$ or $(2cos^2\theta - 1)$

24. $\tan(2\theta) = \frac{2tan\theta}{1-tan^2\theta}$

25. **Product Identities**

$$\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$

$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$

$$\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$$

$$\cos x \sin y = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$$