G. PULLAIAH COLLEGE OF ENGINEERING AND TECHNOLOGY

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Permanently Affiliated to JNTUA, Ananthapuramu

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Department of Electronics & Communication Engineering

Bridge Course

On

PROBABILITY THEORY & STOCHASTIC PROCESSES

By

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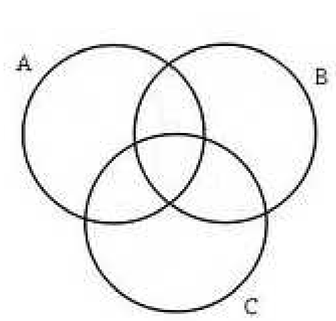
1. Set Theory:

Set Theory is a branch of mathematics which deals with the study of sets or the collection of similar objects.

Set theory is one of the most fundamental branches of mathematics, but is also very complex if you try to analyze three or more sets.

You learn some important set theory formulas in this page which helps you to analyze the group of three or less sets.

If you imagine three sets as:



Set

Note: The sets are overlapping just for the sake of ease, the formulas given here also implies to sets if the overlapped part is null.

Then the following formulas should be correct in the situation:

Finite and Infinite set:

If a set have a finite numbers of elements then a set is called finite set else the set is called infinite set.

For example: D= {0, 2, 4, 6, 8}

A= {a, e ,i, o, u}

The above two sets are finite set while the following sets are infinite one.

The set of stars in Universe. D= {set of natural numbers}

- **Null Set:**

A set which has no element is called Null set. A Null set is also called Empty set or Void set.

It is denoted by symbol ø.

Set Theory Notations & Formulas:

set theory formulas:

$n(A)$– Cardinal number of set A.

$n_o(A)$– Cardinality of set A.

$\bar{A} = A^c$– Complement of set A.

U– Universal set

$A \subset B$– Set A is proper subset of subset of set B.

$A \subseteq B$– Set A is subset of set B.

ϕ– Null set.

$a \in A$– Element "a" belongs to set A.

$A \cup B$– Union of set A and set B.

$A \cap B$– Intersection of set A and set B.

Formulas:

For a group of two sets*:*

1. If A and B are overlapping set, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

2. If A and B are disjoint set, $n(A \cup B) = n(A) + n(B)$

3. $n(A) = n(A \cup B) + n(A \cap B) - n(B)$

4. $n(A \cap B) = n(A) + n(B) - n(A \cup B)$

5. $n(B) = n(A \cup B) + n(A \cap B) - n(A)$

6. $n(U) = n(A) + n(B) - n(A \cap B) + n((A \cup B)^c)$

7. $n((A \cup B)^c) = n(U) + n(A \cap B) - n(A) - n(B)$

8. $n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B)$

9. $n(A - B) = n(A \cup B) - n(B)$

10. $n(A - B) = n(A) - n(A \cap B)$

11. $n(A^c) = n(U) - n(A)$

For a group of three sets:

1. $n(A \cup B \cup C) = n(U) - n((A \cup B \cup C)^c)$

2. $n_0(A) = n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$

3. $n_0(B) = n(B) - n(A \cap B) - n(B \cap C) + n(A \cap B \cap C)$

4. $n_0(C) = n(C) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

5. $n(A \cap Bonly) = n(A \cap B) - n(A \cap B \cap C)$

6. $n(B \cap Conly) = n(B \cap C) - n(A \cap B \cap C)$

7. $n(A \cap Conly) = n(A \cap C) - n(A \cap B \cap C)$

Subsets

When we define a set, if we take pieces of that set, we can form what is called a **subset**.

Example: the set {1, 2, 3, 4, 5}

A **subset** of this is {1, 2, and 3}. Another subset is {3, 4} or even another is {1}, etc.

But {1, 6} is **not** a subset, since it has an element (6) which is not in the parent set.

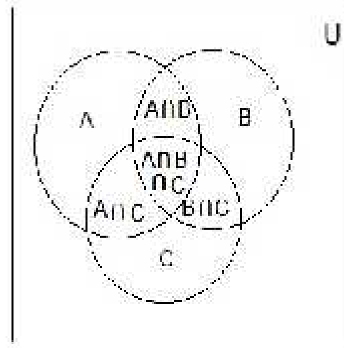
In general: A is a subset of B if and only if every element of A is in B.

2. Venn diagram

Venn diagram is a graphical representation of sets and relation between sets. ***Venn diagram*** is the diagram which shows the possible relations between the finite collections of set. Venn diagram was introduced in 1880 by john Venn.

It is constructed by more than two circles which is generally overlapping. To draw the Venn diagram, you first draw the rectangular which is called "universe" then you draw the required quantity of circle for the collection of set. Whole elements of the set are inside the universe. The elements which are in universe and are not in sets then these elements are place outside the circle but inside the universe. The interior of the circle represent the element of the set while exterior represent the element that are not the member of that sets.

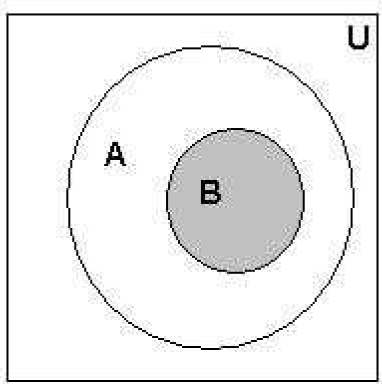
The overlapping area or intersection would represent the common element of the sets. Generally venn diagram contains two and three sets. By using the venn diagram student can easily solve complex problems. Venn diagram can be used to illustrate both sets relationship and logical relationships. It is used in scientific and engineering presentations, in theoretical mathematics, in computer applications, and in statistics. The following venn diagram shows the relationship existing between three sets.



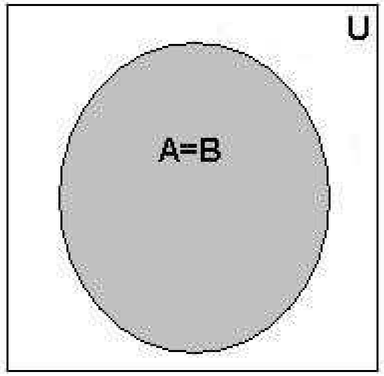
Students can use venn diagram to compare two things, people, place

For example, the order three diagrams consist of three mutually intersecting circles with eight regions. The region A, B and C consists of members which are only in one set not in others. The three regions $A\bigcap B, B\bigcap C, and A\bigcap C$ consist of members which are in two sets but not in three. $A\bigcap B\bigcap C$ Consists of members which are in all three sets and no region occupied represents ϕ.

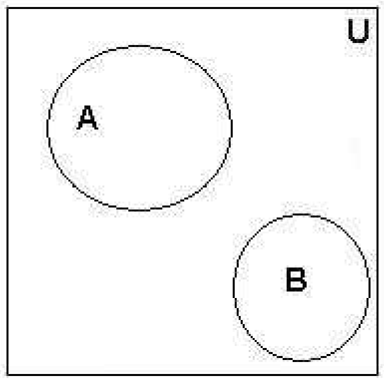
Some of examples of venn diagram showing relation between sets are given below:-



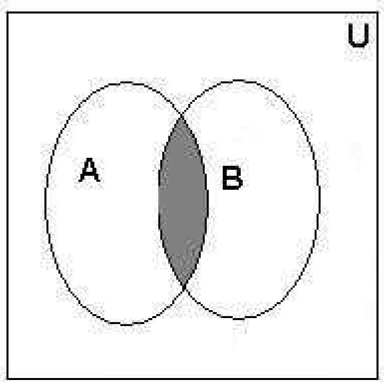
The Venn diagram above represents B⊂A or set B is subset of set A.



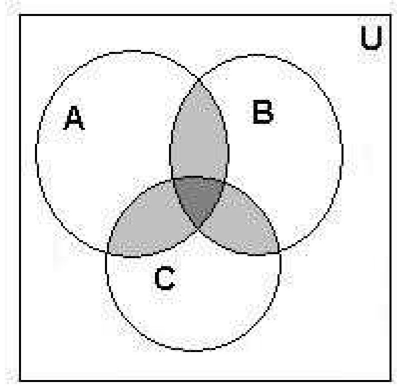
The Venn diagram above represents A=B or set A is equal to set B.



The Venn diagram above represents that set A and set B are disjoint set.



The **Venn** diagram above represents that sets A and B are intersecting.



The **Venn diagram** above represents following things:
a> Set A and set B are intersecting.
b> Set B and set C are intersecting.
c> Set C and set A are intersecting.
d> Set A, B and C are also intersecting.

3. Set Operations:

The Process of making new sets from two or more given sets applying some special rules is known as set operations.
If we are given two sets, then there are three standard ways to construct new sets from them. The three operations are called binary set operations, which are as following:

Union:

A set that contains all the elements contained by first set (A) and second set (B) is known as union of the two sets (A and B).

We denote union of two sets (A and B) by symbol $A \cup B$.

For example: if A={1,2,3} and B={3,4,5} Then, $A \cup B$={1,2,3,4,5}

Intersection:

A set whose elements are the common elements of two sets (A and B) is known as the intersection of the sets (A and B). The intersection of two sets (A and B) is denoted by the symbol $A \cap B$.

For example: If A= {1, 2, 3} and B= {2, 3, 4} Then $A \cap B$= {2, 3}

Complement:

A set whose elements are all the elements of universal set except a set (A) is known as the complement of the set (A). The complement of a set (A) is denoted by symbol Â and read as “A complement”

For example: If A={1,2,3} , B={3,4,5} and C={4,5,6,7} Then , Â={4,5,6,7}

Difference:

The difference of set A and B is the set formed by a set with all elements of set A that does not belongs to set B. We denote the difference of set A and B by A-B and difference if set B and A by B-A.

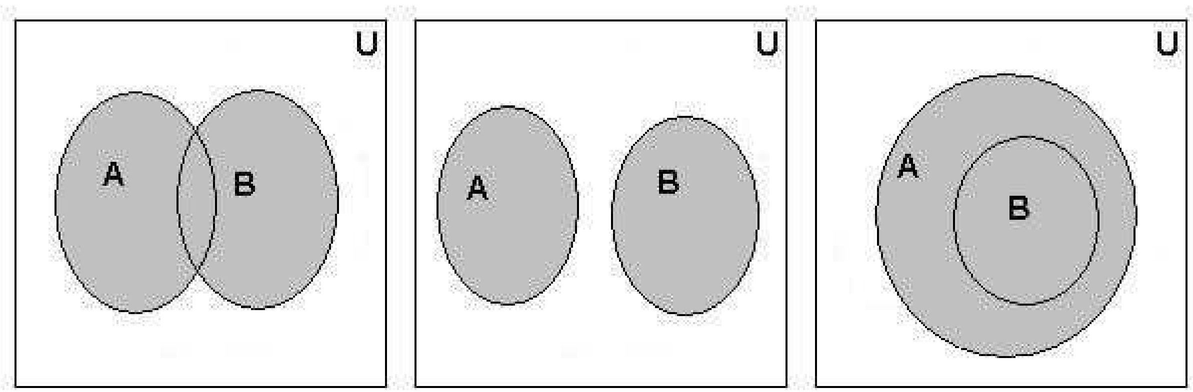
For example: If A= {1, 2, 3, 4} and B= {3, 4, 5, 6} then,

A-B= {1, 2}, B-A= {5, 6}, A-A= ϕ and B-B= ϕ

Above set operations are shown below as graphical representation in Venn diagram.

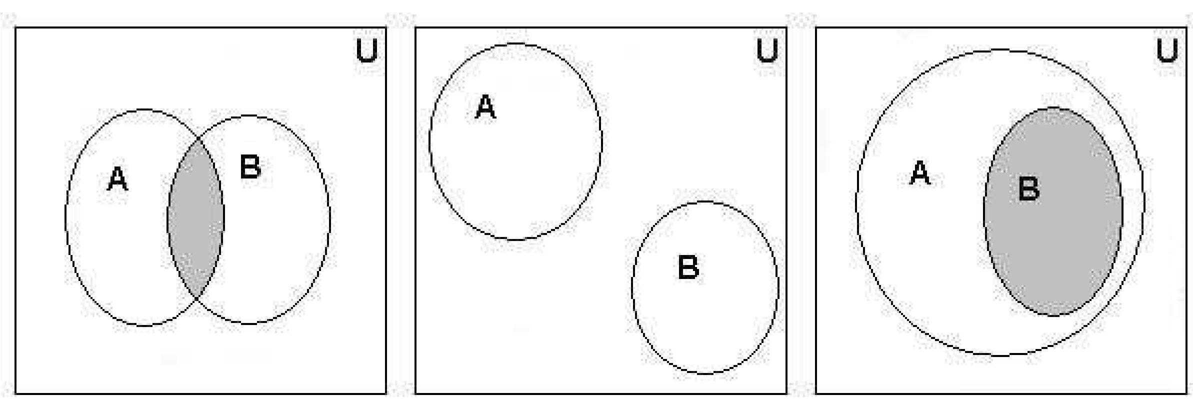
Union:

In the following figures A∪B is shown as shaded region:



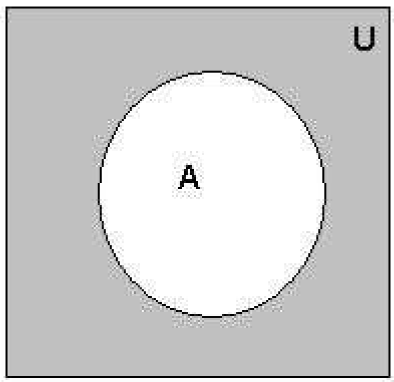
Intersection:

In the following figures A∩B is shown as shaded region, in second figure no region is shaded because in the figure A∩B=Φ



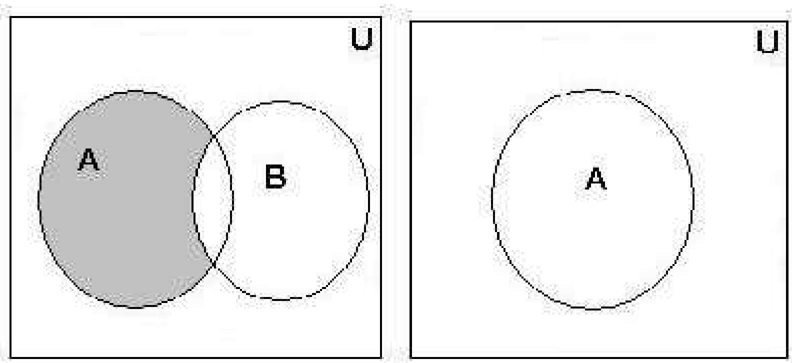
Complement:

In the following figure Â is shown by shaded region:



Difference:

In the first figure below A-B is shown as shaded region and in second figure A-A is shown and no region as shaded as A-A is Φ



4. **FORMULE**:

Differentiation Formulas:

1. $\frac{d}{dx}(x) = 1$
2. $\frac{d}{dx}(ax) = a$
3. $\frac{d}{dx}(x^n) = nx^{n-1}$
4. $\frac{d}{dx}(\cos x) = -\sin x$
5. $\frac{d}{dx}(\sin x) = \cos x$
6. $\frac{d}{dx}(\tan x) = \sec^2 x$
7. $\frac{d}{dx}(\cot x) = -\csc^2 x$
8. $\frac{d}{dx}(\sec x) = \sec x \tan x$
9. $\frac{d}{dx}(\csc x) = -\csc x(\cot x)$
10. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
11. $\frac{d}{dx}(e^x) = e^x$
12. $\frac{d}{dx}(a^x) = (\ln a)a^x$
13. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
14. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
15. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$

Integration Formulas:

1. $\int 1\,dx = x + C$
2. $\int a\,dx = ax + C$
3. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$
4. $\int \sin x\,dx = -\cos x + C$
5. $\int \cos x\,dx = \sin x + C$
6. $\int \sec^2 x\,dx = \tan x + C$
7. $\int \csc^2 x\,dx = -\cot x + C$
8. $\int \sec x(\tan x)\,dx = \sec x + C$
9. $\int \csc x(\cot x)\,dx = -\csc x + C$
10. $\int \frac{1}{x}\,dx = \ln|x| + C$
11. $\int e^x\,dx = e^x + C$
12. $\int a^x\,dx = \frac{a^x}{\ln a} + C\ a > 0,\ a \neq 1$
13. $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$
14. $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$
15. $\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \sec^{-1} x + C$

5. PROBABILITY Introduction:

The basic to the study of probability is the idea of a Physical experiment. A single performance of the experiment is called a trial for which there is an outcome. Probability can be defined in three ways.

1. The First one is Classical Definition.
2. Second one is Definition from the knowledge of Sets Theory and Axioms.
3. And the last one is from the concept of relative frequency.

Experiment: Any physical action can be considered as an experiment.
Tossing a coin, Throwing or rolling a die or dice and drawing a card from a deck of 52-cards are Examples for the Experiments.

Sample Space: The set of all possible outcomes in any Experiment is called the sample space and it is represented by the letter 's'. The sample space is a universal set for the experiment.
The sample space can be of 4 types. They are:
1. Discrete and finite sample space.
2. Discrete and infinite sample space.
3. Continuous and finite sample space.
4. Continuous and infinite sample space.

- Tossing a coin, throwing a dice are the examples of discrete finite sample space.
- Choosing randomly a positive integer is an example of discrete infinite sample space.
- Obtaining a number on a spinning pointer is an example for continuous finite sample space.
- Prediction or analysis of a random signal is an example for continuous infinite sample space.

Event: An event is defined as a subset of the sample space. The events can be represented with capital letters like A, B, C etc… All the definitions and operations applicable to sets will apply to events also.
As with sample space events may be of either discrete or continuous. Again the in discrete and continuous they may be either finite or infinite. If there are N numbers of elements in the sample space of an experiment then there exists 2N number of events.
The event will give the specific characteristic of the experiment whereas the sample space gives all the characteristics of the experiment.

Classical Definition: From the classical way the probability is defined as the ratio of number of favorable outcomes to the total number of possible outcomes from an experiment. I.e. Mathematically,

$$P(A) = F/T.$$

Where: P (A) is the probability of event A.
F is the number of favorable outcomes and
T is the Total number of possible outcomes.
The classical definition fails when the total number of outcomes becomes infinity.

Definition from Sets and Axioms: In the axiomatic definition, the probability P(A) of an event is always a non-negative real number which satisfies the following three Axioms.
Axiom 1: $P(A) \geq 0$.Which means that the probability of event is always a non-negative number
Axiom 2: $P(S) = 1$.Which means that the probability of a sample space consisting of all possible outcomes of experiment is always unity or one.
Axiom 3: P (U (n=1 to N)) or P (A1 A2 . . . AN) = P (A1) + P (A2) + . . . + P (AN)
This means that the probability of Union of N number of events is same as the Sum of the individual probabilities of those N Events.

Probability as a relative frequency: The use of common sense and engineering and scientific Observations lead to a definition of probability as a relative frequency of occurrence of some event. Suppose that a random experiment repeated n times and if the event A occurs n(A) times, then the probability of event a is defined as the relative frequency of event a when the number of trials n tends to infinity. Mathematically P (A) =Lt n->∞ n (A)/n

Where n (A)/n is called the relative frequency of event, A.

Mathematical Model of Experiments: Mathematical model of experiments can be derived from the axioms of probability introduced. For a given real experiment with a set of possible outcomes, the mathematical model can be derived using the following steps:

1. Define a sample space to represent the physical outcomes.
2. Define events to mathematically represent characteristics of favorable outcomes.
3. Assign probabilities to the defined events such that the axioms are satisfied.

Multiplication Theorem of Probability: Multiplication theorem can be used to find out probability of outcomes when an experiment is performing on more than one event. It states that if there are N events An, n=1,2, . . . N, in a given sample space, then the joint probability of all the events can be expressed as And if all the events are independent, then

Permutations & Combinations: An ordered arrangement of events is called Permutation. If there are n numbers of events in an experiment, then we can choose and list them in order by two conditions. One is with replacement and another is without replacement. In first condition, the first event is chosen in any of the n ways thereafter the outcome of this event is replaced in the set and another event is chosen from all v events. So the second event can be chosen again in n ways. For choosing r events in succession, the numbers of ways are n_r. In the second condition, after choosing the first event, in any of the n ways, the outcome is not replaced in the set so that the second event can be chosen only in (n-1) ways. The third event in (n-2) ways and the r^{th} event in (n-r+1) ways. Thus the total numbers of ways are n (n-1) (n-2) . . . (n-r+1).

Problems:

1: Say, a coin is tossed twice. What is the probability of getting two consecutive tails?

Solution:

Probability of getting a tail in one toss = 1/2

The coin is tossed twice. So 1/2 * 1/2 = 1/4 is the answer.

Here's the verification of the above answer with the help of sample space.

When a coin is tossed twice, the sample space is {(H,H), (H,T), (T,H), (T,T)}.

Our desired event is (T, T) whose occurrence is only once out of four possible outcomes and hence, our answer is 1/4.

2: Consider another example where a pack contains 4 blue, 2 red and 3 black pens. If a pen is drawn at random from the pack, replaced and the process repeated 2 more times, what is the probability of drawing 2 blue pens and 1 black pen?

Solution:

Here, total number of pens = 9
Probability of drawing 1 blue pen = 4/9
Probability of drawing another blue pen = 4/9
Probability of drawing 1 black pen = 3/9
Probability of drawing 2 blue pens and 1 black pen = 4/9 * 4/9 * 3/9 = 8/81

3: A pack contains 4 blue, 2 red and 3 black pens. If 2 pens are drawn at random from the pack, NOT replaced and then another pen is drawn. What is the probability of drawing 2 blue pens and 1 black pen?

Solution: Probability of drawing 1 blue pen = 4/9
Probability of drawing another blue pen = 3/8
Probability of drawing 1 black pen = 3/7
Probability of drawing 2 blue pens and 1 black pen = 4/9 * 3/8 * 3/7 = 1/14

4: What is the probability of drawing a king and a queen consecutively from a deck of 52 cards, without replacement?

Solution: Probability of drawing a king = 4/52 = 1/13
After drawing one card, the numbers of cards are 51.
Probability of drawing a queen = 4/51.
Now, the probability of drawing a king and queen consecutively is 1/13 * 4/51 = 4/663
5. The sample space S of two dice is shown below.

S = { (1,1),(1,2),(1,3),(1,4),(1,5),(1,6)
(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)
(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)
(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)
(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)
(6,1),(6,2),(6,3),(6,4),(6,5),(6,6) }

a)Let E be the event "sum equal to 1". There are no outcomes which correspond to a sum equal to 1, hence

P(E) = n(E) / n(S) = 0 / 36 = 0

b) Three possible outcomes give a sum equal to 4: E = {(1,3),(2,2),(3,1)}, hence.

P(E) = n(E) / n(S) = 3 / 36 = 1 / 12

c) All possible outcomes, E = S, give a sum less than 13, hence.

P(E) = n(E) / n(S) = 36 / 36 = 1.