UNIT – II

TIME RESPONSE ANALYSIS

The transient response and steady state behaviour of a system are together referred to as time response analysis.

The behaviour of a system from initial state to final state is referred to as transient response. The behaviour of a system as time 't' tends to infinity is referred to as steady-state response. Thus, the system response c(t) may be written as

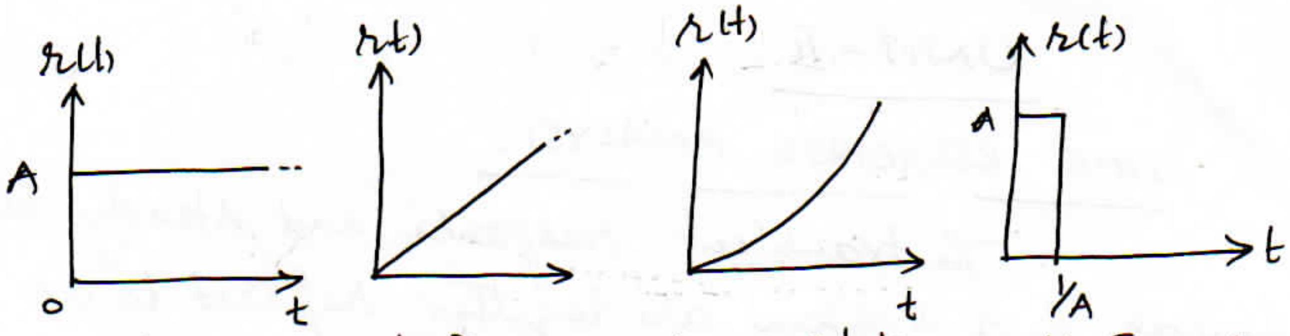
$$c(t) = c_{tr}(t) + c_{ss}(t)$$

The transient response is the response of the system when the input changes from one state to another. The steady state response is the response as the time 't' approaches infinity.

Standard Test Signals: The knowledge of input signal is required to predict the response of a system. The characteristics of actual input signals are sudden shock, sudden change, constant velocity and constant acceleration. Test signals with these characteristics are used as input signals to predict the performance of the system. The commonly used test input signals are step, ramp, parabolic and impulse.

(1) **Step Signal**: The step is a signal whose value changes from one level to another level in zero time. The mathematical representation of the step function is

$$r(t) = A\,u(t)\ ; \quad \text{where } u(t) = 1 \quad t > 0$$
$$= 0 \quad t < 0$$



(a) Step (b) Ramp (c) parabolic (d) Impulse

Figure: Standard test signals

In Laplace transform form $R(s) = \frac{A}{s}$

(2) Ramp Signal: The ramp is a signal which starts at a value of zero and increases linearly with time.

$$r(t) = At; \quad t > 0$$
$$= 0 \quad t < 0$$

In the Laplace transform form $R(s) = \frac{A}{s^2}$

The ramp is integral of step signal.

(3) parabolic Signal: The parabolic signal is the integral of ramp signal and is given by

$$r(t) = \frac{At^2}{2}; \quad t > 0$$
$$= 0 \quad t < 0$$

In the Laplace transform form $R(s) = A/s^3$

(4) Impulse Signal: An impulse is a signal whose value is zero every where except at $t=0$. At $t=0$ it has an infinite magnitude

$$\delta(t) = 0; \quad t \neq 0$$
$$\lim_{\epsilon \to 0} \int_{\epsilon^-}^{\epsilon^+} \delta(t)\,dt = 1$$

Since a perfect impulse can not be achieved in practice, it is usually approximated by a pulse of small width but unit area.

Impulse is the derivative of Step Signal

$$\delta(t) = \frac{d}{dt}u(t) = \dot{u}(t)$$

$$L[\delta(t)] = 1$$

Let us consider a system with transfer function

$$\frac{C(s)}{R(s)} = G(s) \; : \; \text{if input } r(t) = \delta(t) \text{ then } R(s) = 1$$

$$\therefore C(s) = G(s)R(s) = G(s)$$

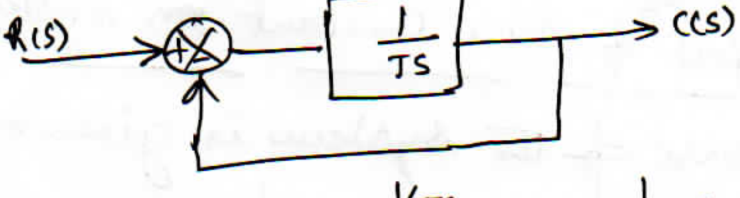
taking inverse LT on both sides

$$\boxed{c(t) = g(t)}$$

Thus, the impulse response of a system, indicated by $g(t)$ is the inverse Laplace transform of its transfer function $G(s)$. This is sometimes referred to as weighting function of the system. The weighting function of a system can be used to find the system's response to any input $r(t)$ by means of the convolution integral. Thus

$$c(t) = \int_0^t g(t-\tau)\, r(\tau)\, d\tau$$

Time Response of First-order Systems: Let us consider a first order system with unity feedback shown in figure.



$$\therefore \frac{C(s)}{R(s)} = \frac{1/TS}{\frac{1}{TS}+1} = \frac{1}{1+TS}$$

(i) Response to the unit-step input: if $r(t) = u(t)$, then $R(s) = \frac{1}{s}$

$$\therefore C(s) = R(s)\cdot\frac{1}{1+TS} = \frac{1}{s(1+Ts)} = \frac{1}{s} - \frac{T}{Ts+1}$$

Taking inverse Laplace transform

$$c(t) = 1 - e^{-t/T}$$

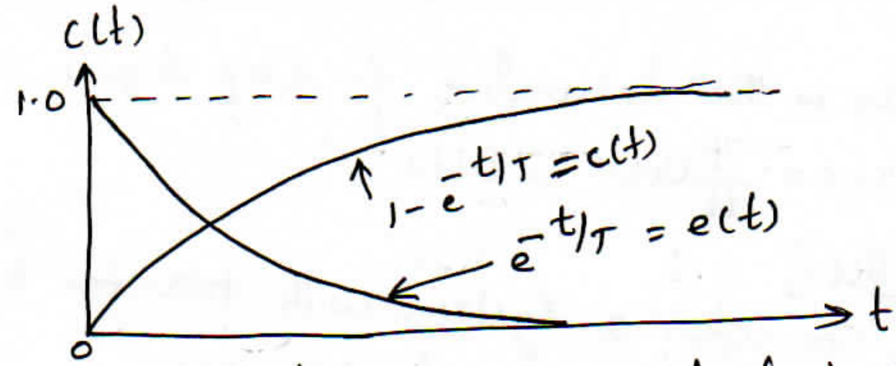


Figure : Unit-step response of first-order system

It is seen that the output rises exponentially from zero value to final value of unity.

The initial slope of the curve at $t=0$ is given by

$$\frac{d}{dt}c(t)\Big|_{t=0} = \frac{1}{T}e^{-t/T}\Big|_{t=0} = \frac{1}{T}$$

where 'T' is known as the time constant of the system

The time constant is indicative of how fast the system tends to reach the final value. A large time constant corresponds to a sluggish system and a small time constant corresponds to a fast response

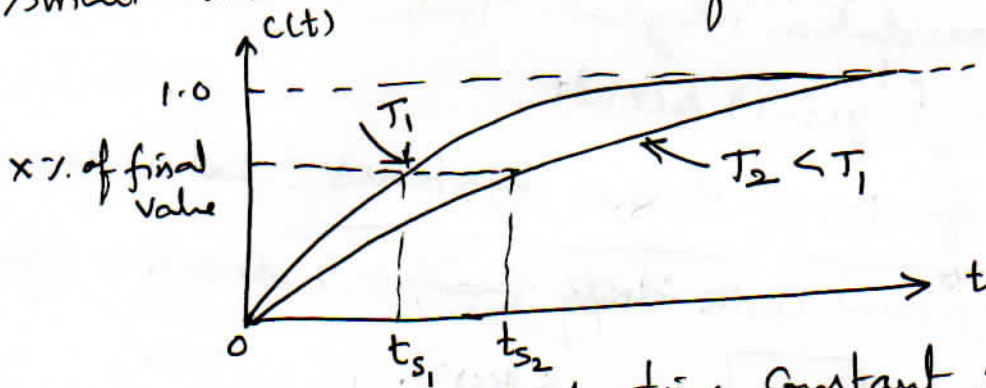


Figure : Effect of time constant on system response

The error response of the system is given by

$$e(t) = r(t) - c(t) = e^{-t/T}$$

The steady state error e_{ss} is given by

$$e_{ss} = \underset{t\to\infty}{Lt}\ e(t) = \underset{s\to 0}{Lt}\ sE(s) = 0$$

Thus this first order system tracks the unit-step input with zero steady state error.

(2) Response to the unit-ramp input:

if $r(t) = t$; then $R(s) = 1/s^2$

$$\frac{C(s)}{R(s)} = \frac{1}{Ts+1} \quad \therefore \; C(s) = R(s)\cdot\frac{1}{Ts+1}$$

or $$C(s) = \frac{1}{s^2(Ts+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{sT+1}$$
$$= -\frac{T}{s} + \frac{1}{s^2} + \frac{T^2}{Ts+1}$$

taking inverse Laplace transform

$$c(t) = -T + t + Te^{-t/T}$$

$\therefore$ The error signal $e(t) = r(t) - c(t) = T(1-e^{-t/T})$

The steady state error $e_{ss} = \lim_{t\to\infty} e(t) = T$
$$= \lim_{s\to 0} sE(s) = T$$

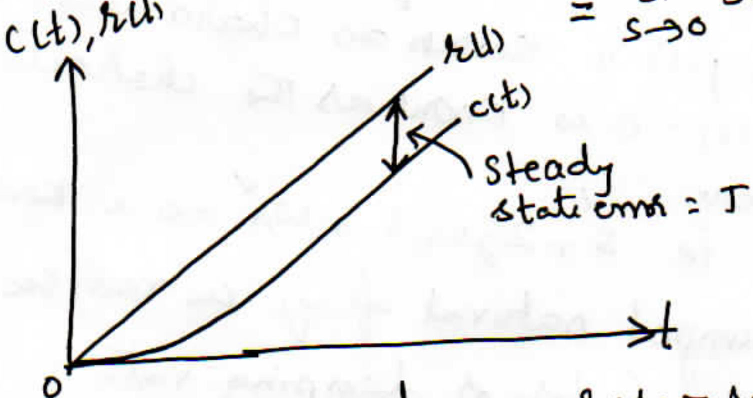


Figure: Unit ramp response of first order system

Thus the first order system will track the unit-ramp input with a steady state error T, which is equal to the time constant of the system.

By reducing the system time constant, we can improve the speed of the response but also reduces the steady-state error to a ramp input.

Time Response of 2nd order System :

Let us consider a second order system shown in figure.

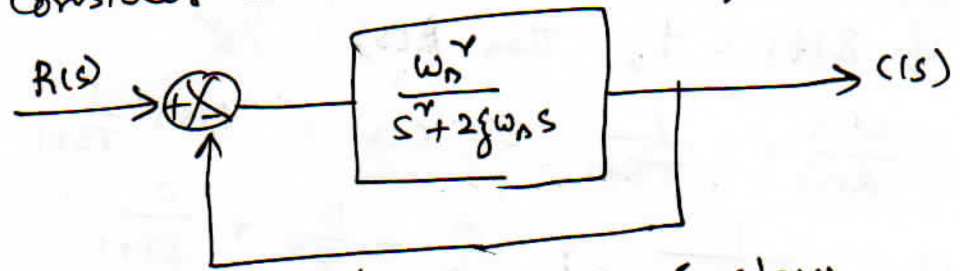


Figure : unity feedback system

The transfer function $\frac{C(s)}{R(s)} = \frac{\left(\omega_n^2 / s^2+2\xi\omega_n s\right)}{1+\frac{\omega_n^2}{s^2+2\xi\omega_n s}(1)}$

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \longrightarrow (1)$$

$$= \frac{p(s)}{q(s)}$$

where the denominator $q(s)$ is known as characteristic polynomial and $q(s)=0$ is known as the characteristic equation of the system. ie $s^2+2\xi\omega_n s+\omega_n^2=0$ is known as CE

where ω_n = undamped natural freq in rad/sec

ξ = damping factor or damping ratio

if $\xi=0$, undamped system

$0<\xi<1$, under damped system

$\xi=1$, critically damped system

$\xi>1$, over damped system

$\omega_d=\omega_n\sqrt{1-\xi^2}$ is damped natural frequency.

Unit step response of 2nd order system : If $r(t)=u(t)$, then $R(s)=\frac{1}{s}$ $\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

and $C(s) = R(s)\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

or $C(s) = \frac{1}{s}\frac{\omega_n^2}{(s^2+2\xi\omega_n s+\omega_n^2)}$

$$= \frac{1}{s} - \frac{s+2\xi\omega_n}{(s+\xi\omega_n)^2+\omega_n^2(1-\xi^2)}$$

let $\omega_d = \omega_n\sqrt{1-\xi^2}$

$$\therefore C(s) = \frac{1}{s} - \frac{s+2\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}$$

$$= \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}$$

Taking inverse LT on both sides

$$L^{-1}C(s) = c(t) = L^{-1}\left\{\frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{\omega_d}\cdot\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}\right\}$$

$$\therefore c(t) = 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi\omega_n}{\omega_d}e^{-\xi\omega_n t}\sin\omega_d t$$

put $= \omega_d = \omega_n\sqrt{1-\xi^2}$

$$\therefore c(t) = 1 - e^{-\xi\omega_n t}\left\{\cos\omega_d t + \frac{\xi\cancel{\omega_n}}{\cancel{\omega_n}\sqrt{1-\xi^2}}\sin\omega_d t\right\}$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left\{\sqrt{1-\xi^2}\cos(\omega_d t) + \xi\sin(\omega_d t)\right\}$$

let $\xi = \cos\phi$ $\therefore$ $\sin\phi = \sqrt{1-\cos^2\phi} = \sqrt{1-\xi^2}$

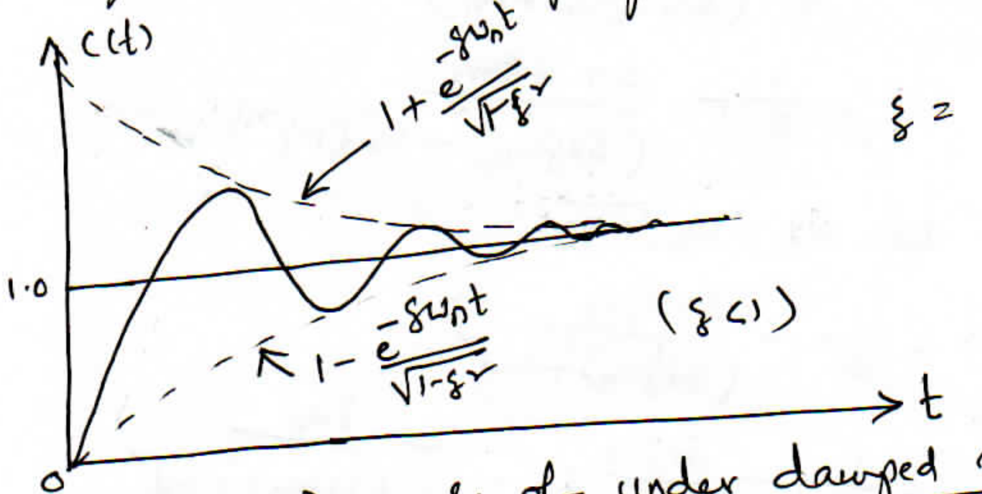
$$\therefore c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left\{\sin\phi\cos(\omega_d t) + \cos\phi\sin(\omega_d t)\right\}$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t+\phi)$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin\left[(\omega_n\sqrt{1-\xi^2})t + \tan^{-1}\sqrt{\frac{1-\xi^2}{\xi}}\right]$$

The steady state value of $c(t) = \lim_{t\to\infty} c(t) = 1$

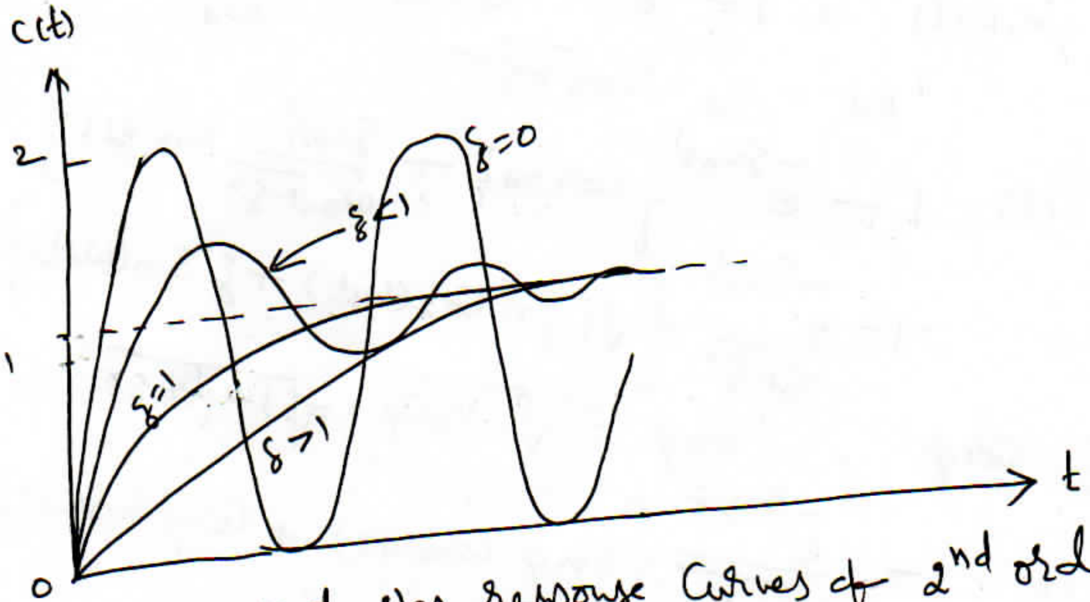
The time response of under damped ($\xi<1$) second order system for unit step input is shown in figure



Figure(1): Time Response of under damped second order system for unit step input.

If $\xi=0$, $c(t) = 1-\cos\omega_n t$

$\xi=1$ $\quad c(t) = 1-e^{-\omega_n t} - \omega_n t\, e^{-\omega_n t}$



Figure(2): Unit step response curves of 2nd order system for different values of 'ξ'.

The characteristic equation of 2nd order system is

$$s^2+2\xi\omega_n s+\omega_n^2=0$$

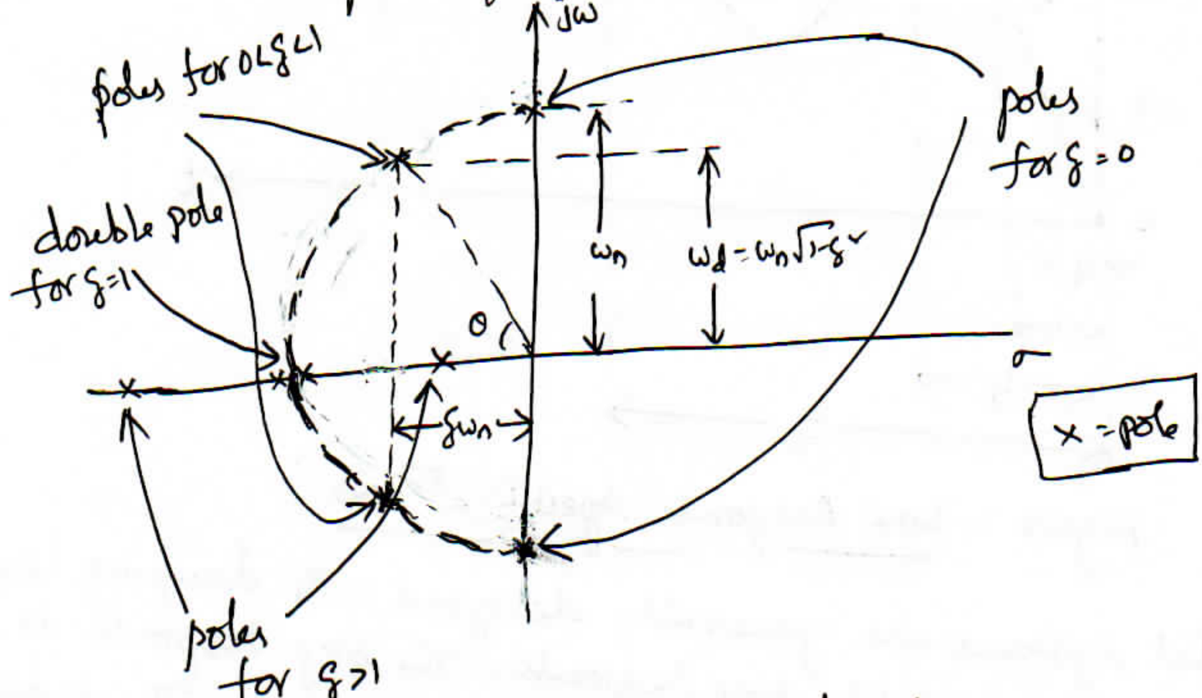
The roots of CE are given by

$$s_1, s_2 = \frac{-2\xi\omega_n \pm \sqrt{(2\xi\omega_n)^2-4\omega_n^2}}{2}$$

$$s_1, s_2 = \frac{-2\zeta\omega_n \pm \sqrt{-1}\sqrt{4}\sqrt{\omega_n^2 - \zeta^2\omega_n^2}}{2}$$

$$= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

Important: The roots of the characteristic equation are the poles of closed loop system



Figure(3): pole locations of 2nd order system for different values of 'ζ'

For $\zeta = 0$, the poles lie on the imaginary axis

$0 < \zeta < 1$, the poles are Complex Conjugate and lie in LHS

$\zeta = 1$, Double pole on the Real axis in LHS

for $\zeta > 1$, the poles move in opposite direction on the Real axis

Time - Domain Specifications :

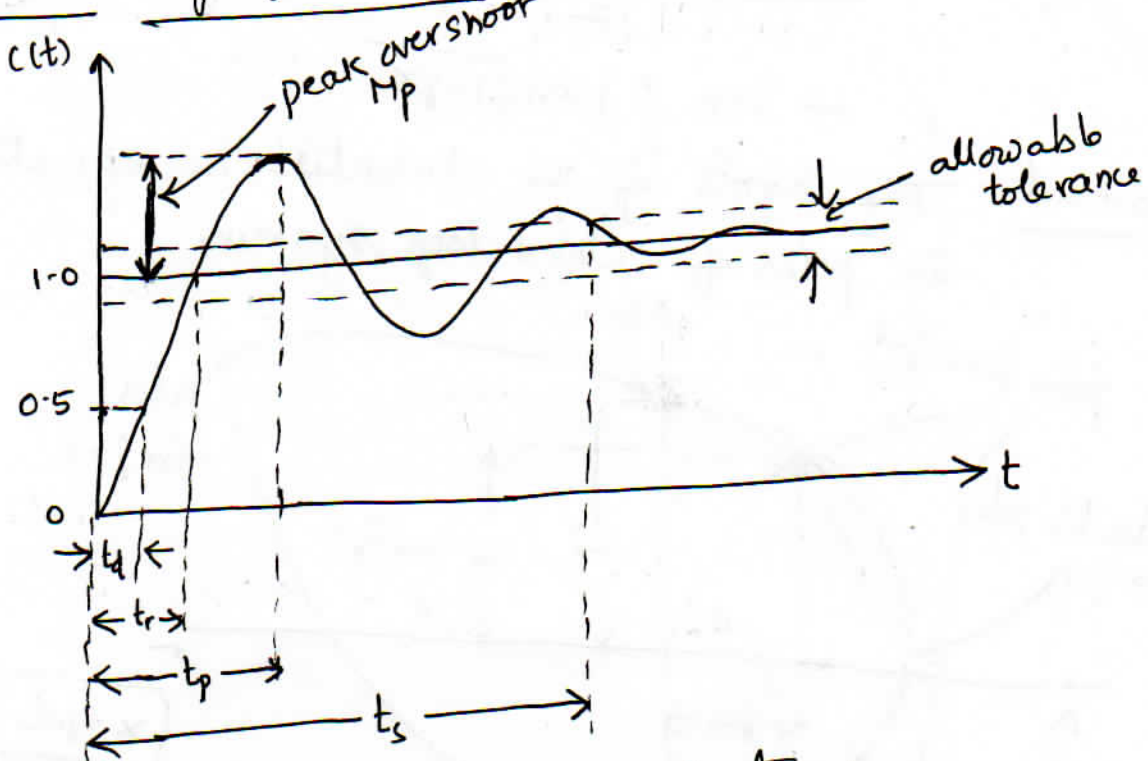


Figure : Time Response Specifications

Control systems are generally designed with damping less than one; ie, oscillatory step response. The step response is characterized by the following performance indices. The indices are qualitatively related to

(i) How fast the system moves to follow the input ?
(ii) How oscillatory it is ($\xi = ?$) ?
(iii) How long does it take to practically reach the final value?

It may be noted that various indices are not independent of each other.

(1) Delay time(t_d): It is the time required for the response to reach 50% of the final value in first attempt.

(2) Rise time (t_r) : It is the time required for the response to rise from 10% to 90% of the final value for over damped systems and 0 to 100% of the final value for under damped systems

(3) peak time (t_p) : It is the time required for the response to reach the peak of time response or the peak over shoot.

(4) Peak over shoot (M_p): It indicates the normalized difference between the time response peak and the steady output and is defined as

$$\%\ \text{peak over shoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100$$

(5) Settling time (t_s): It is the time required for the response to reach and stay within a specified tolerance band (usually 2% or 5%) of its final value.

(6) Steady state error (e_{ss}): It indicates the error between the actual output and desired output as 't' tends to infinity

ie $e_{ss} = \lim_{t\to\infty} [r(t) - c(t)]$

Time Response Specifications of Second-order Systems:

The unit step response of 2nd order system is given by

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin[\omega_d t + \phi] \longrightarrow ①$$

where $\omega_d = \omega_n\sqrt{1-\xi^2}$; $\phi = \tan^{-1}\sqrt{\frac{1-\xi^2}{\xi}}$

(1) Rise time (t_r): The rise-time t_r is obtained when $c(t)$ reaches unity ie $c(t)\big|_{t=t_r} = 1$

$$\Rightarrow c(t_r) = 1 - \frac{e^{-\xi\omega_n t_r}}{\sqrt{1-\xi^2}} \sin[\omega_d t_r + \phi] = 1$$

$\sin\theta = 0$; for $\theta = n\pi$, $n = 0, 1, 2 \ldots$

$$\Rightarrow \sin(\omega_d t_r + \phi) = 0$$

$\therefore \omega_d t_r + \phi = \pi$; (before completing one cycle)

$$\text{or } t_r = \frac{\pi - \phi}{\omega_d} = \frac{\pi - \tan^{-1}\sqrt{1-\xi^2}/\xi}{\omega_n\sqrt{1-\xi^2}} \longrightarrow (i)$$

(2) peak time (t_p): up to peak time 't_p' the response $c(t)$ increases, then decreases.

$$\therefore \frac{d}{dt} c(t)\Big|_{t=t_p} = 0$$

$$\frac{d}{dt}\left\{1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \phi)\right\}\Big|_{t=t_p} = 0$$

$$\left[-\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \cos(\omega_d t + \phi)\,\omega_d - \sin(\omega_d t + \phi) \cdot \frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \cdot (-\xi\omega_n)\right]\Big|_{t=t_p} = 0$$

or $\sin(\omega_d t_p + \phi)\,\xi - \cos(\omega_d t_p + \phi)\sqrt{1-\xi^2} = 0$

where $\xi = \cos\phi \quad \therefore \sin\phi = \sqrt{1-\xi^2}$

$$\sin(\omega_d t_p + \phi)\cos\phi - \cos(\omega_d t_p + \phi)\sin\phi = 0$$

$$\sin(\omega_d t_p + \phi - \phi) = 0$$

or $\sin(\omega_d t_p) = 0$

or $\omega_d t_p = n\pi$

since the first peak occurs before 2π

$$\therefore \omega_d t_p = \pi \longrightarrow \text{(ii)}$$

or $t_p = \dfrac{\pi}{\omega_d} = \dfrac{\pi}{\omega_n\sqrt{1-\xi^2}}$

This is the time required to reach first peak overshoot
The first undershoot occurs at $t = \dfrac{2\pi}{\omega_n\sqrt{1-\xi^2}}$ and the 2nd over shoot occurs at $t = \dfrac{3\pi}{\omega_n\sqrt{1-\xi^2}}$

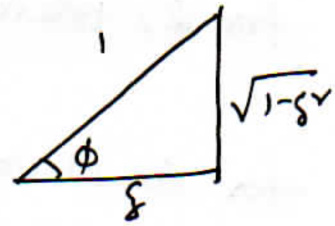
(3) peak over shoot (M_p): $M_p = c(t_p) - 1$

$$= 1 - \frac{e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}} \sin(\omega_d t_p + \phi) - 1$$

$$= -\frac{e^{-\xi\omega_n \frac{\pi}{\omega_n\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \sin\left[\omega_n\sqrt{1-\xi^2}\cdot\frac{\pi}{\omega_n\sqrt{1-\xi^2}} + \phi\right] \quad (7)$$

$$= -e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} \sin(\pi+\phi)\frac{1}{\sqrt{1-\xi^2}}$$

$$= -e^{\frac{-\pi\xi}{\sqrt{1-\xi}}}\left[-\sin\phi\right]$$



$$= +e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}\sqrt{1-\xi^2}\cdot\frac{1}{\sqrt{1-\xi^2}}$$

$$= e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}$$

% peak overshoot $= 100\, e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}$

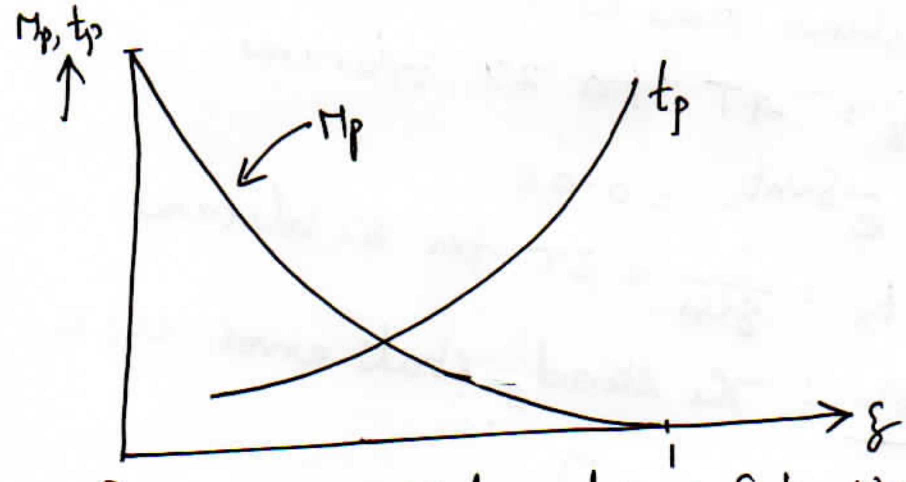


Figure : variation of M_p & t_p w.r.t ξ

From the figure, it is observed that M_p decreases and 't_p' increases as ξ increases.

(4) Settling time (t_s): The response of 2nd order system has two components (i) Decaying exponential Component $\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}$

(ii) Sinusoidal Component $\sin(\omega_d t + \theta)$

The decaying exponential term reduces the oscillations produced by sinusoidal Component. Hence, the settling is decided by exponential component. The settling time can be found by equating exponential component to percentage of tolerance.

for 2% tolerance, $\left.\dfrac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right|_{t=t_s} = 0.02$

for small values of ζ $\sqrt{1-\zeta^2} \simeq 1$

$\therefore$ $e^{-\zeta\omega_n t_s} = 0.02$

taking 'ln' on both sides

$-\zeta\omega_n t_s = \ln(0.02) = -4$

or $t_s = \dfrac{4}{\zeta\omega_n}$ for 2% tolerance

for second order systems time constant $T = \dfrac{1}{\zeta\omega_n}$

$\therefore$ settling $t_s = 4T$ for 2% tolerance

For 5% tolerance $e^{-\zeta\omega_n t_s} = 0.05$

$\therefore t_s = \dfrac{3}{\zeta\omega_n} = 3T$ for 5% tolerance

(5) Steady state Error : The steady state error e_{ss} is given by $e_{ss} = \lim\limits_{t\to\infty}\left[r(t) - c(t)\right]$

(i) for unit step input $r(t) = u(t) = 1$

$\therefore e_{ss} = \lim\limits_{t\to\infty}\left[1 - c(t)\right] = 0$

Thus, the second order system has zero steady state error to unit step input.

(ii) for ramp input $r(t) = t$

$\therefore$ steady state error $= \lim\limits_{t\to\infty}\left[t - c(t)\right] = \lim\limits_{s\to 0} s\left[\dfrac{1}{s^2} - C(s)\right]$

$= \dfrac{2\zeta}{\omega_n}$

Steady state Errors & Error Constants :

The steady state error is a measure of system accuracy. These errors arise from the nature of inputs, type of system and from non-linearities of system components such as static friction, backlash etc.

Let us consider a feed back system shown in figure.

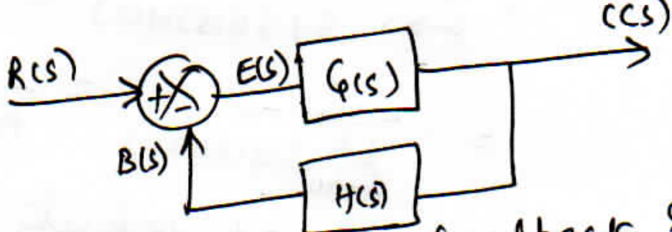


Figure: Negative feedback system

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} \quad \text{and} \quad C(s) = E(s)G(s)$$

$E(s)$ = Error Signal

$$\therefore E(s) = \frac{C(s)}{R(s)} = \frac{R(s)}{1+G(s)H(s)}$$

$$\therefore \text{The steady state error } e_{ss} = \underset{s\to 0}{Lt}\; sE(s)$$

$$= \underset{s\to 0}{Lt}\; s\frac{R(s)}{1+G(s)H(s)} \longrightarrow (1)$$

The above expression shows that the steady state error depends upon type of input and forward path transfer function G(s). The steady state errors for various types of standard input signals are derived below

(1) Unit-step input : $r(t) = u(t) \quad \therefore R(s) = \frac{1}{s}$

$\therefore$ For unit step input, $e_{ss} = \underset{s\to 0}{Lt}\; \dfrac{s\cdot\frac{1}{s}}{1+G(s)H(s)}$

$$= \underset{s\to 0}{Lt}\; \frac{1}{1+G(s)H(s)}$$

$$= \frac{1}{1+\underset{s\to 0}{Lt}\, G(s)H(s)} = \frac{1}{1+K_p}$$

where $K_p = \underset{s\to 0}{Lt}\, G(s)H(s)$ is defined as position error constant

(2) For unit-ramp (velocity) input:

$$r(t) = t \quad \therefore R(s) = \frac{1}{s^2}$$

Steady state error $e_{ss} = \underset{s \to 0}{Lt} \frac{s R(s)}{1+G(s)H(s)}$

$$= \underset{s \to 0}{Lt} \frac{s \cdot \frac{1}{s^2}}{1+G(s)H(s)}$$

$$= \underset{s \to 0}{Lt} \frac{1}{s + sG(s)H(s)}$$

$$= \frac{1}{\underset{s \to 0}{Lt}\, sG(s)H(s)} = \frac{1}{K_v}$$

where $K_v = \underset{s \to 0}{Lt}\, sG(s)H(s)$ is known as velocity error constant

(3) unit-parabolic (acceleration) input:

For parabolic input $r(t) = t^2/2 \quad \therefore R(s) = \frac{1}{s^3}$

$\therefore$ The steady state error $e_{ss} = \underset{s \to 0}{Lt} \frac{s(1/s^3)}{1+G(s)H(s)}$

$$= \underset{s \to 0}{Lt} \frac{1}{s^2 + s^2 G(s)H(s)}$$

$$= \frac{1}{\underset{s \to 0}{Lt}\, s^2 G(s)H(s)} = \frac{1}{K_a}$$

where $K_a = \underset{s \to 0}{Lt}\, s^2 G(s)H(s)$ is known as acceleration error constant.

Types of Feedback Control systems: The openloop transfer function $G(s)$ can be written in two standard forms namely time-constant form and pole-zero form.

$$G(s) = \frac{K(T_{z_1}s+1)(T_{z_2}s+1)\cdots}{s^n(T_{p_1}s+1)(T_{p_2}s+1)\cdots} \qquad \text{Time-Constant form}$$

$$= \frac{K'(s+z_1)(s+z_2)\cdots}{s^n(s+p_1)(s+p_2)\cdots} \qquad \text{pole-zero form}$$

The term s^n corresponds to number of integrations in the system. s^n also represents number of poles at the origin. The number of poles at the origin is also known as the type of system. Now, we can determine steady state errors for different types of systems.

(1) Type-0 Systems: If $n=0$, the steady state errors to various inputs are as follows $G(s) = \frac{K(s+z_1)(s+z_2)\cdots}{(s+p_1)(s+p_2)\cdots}$

(i) $e_{ss}(\text{position}) = \frac{1}{1+\lim_{s\to 0} G(s)H(s)} = \frac{1}{1+K_p}$ $\boxed{\text{If } H(s)=1}$

(ii) $e_{ss}(\text{velocity input}) = \frac{1}{\lim_{s\to 0} sG(s)H(s)}$

$= \frac{1}{\lim_{s\to 0} sG(s)H(s)} = \infty$ $\boxed{\text{If } H(s)=1}$

(iii) $e_{ss}(\text{acceleration}) = \frac{1}{\lim_{s\to 0} s^2 G(s)H(s)} = \frac{1}{0} = \infty$ If $H(s)=1$

Thus a type-0 system has constant position error, infinite velocity and acceleration errors.

(2) Type-1 System: If $n=1$; $G(s) = \frac{K(s+z_1)(s+z_2)\cdots}{s(s+p_1)(s+p_2)\cdots}$

(i) $e_{ss}(\text{for position input}) = \frac{1}{1+\lim_{s\to 0} G(s)H(s)}$ if $H(s)=1$

$$e_{ss} = \frac{1}{1+\lim_{s\to 0} \frac{(s+z_1)(s+z_2)\cdots}{s(s+p_1)(s+p_2)\cdots}}$$

$$= \frac{1}{1+\infty} = 0$$

(ii) $e_{ss}(\text{velocity input}) = \dfrac{1}{\underset{s\to 0}{Lt}\ sG(s)H(s)}$

for unity feedback system $H(s) = 1$

$\therefore e_{ss}(\text{velocity}) = \dfrac{1}{\underset{s\to 0}{Lt}\ \cancel{s}\ \dfrac{K'(s+z_1)(s+z_2)\cdots}{\cancel{s}(s+p_1)(s+p_2)\cdots}}$

$= \dfrac{1}{K'} = \dfrac{1}{K_v}$

(iii) $e_{ss}(\text{acceleration input}) = \dfrac{1}{\underset{s\to 0}{Lt}\ s^2 G(s)H(s)}$

for unity feedback system $H(s) = 1$

$\therefore e_{ss}(\text{acceleration}) = \dfrac{1}{\underset{s\to 0}{Lt}\ s^{\cancel{2}}\cdot \dfrac{K'(s+z_1)(s+z_2)\cdots}{\cancel{s}(s+p_1)(s+p_2)\cdots}}$

$= \dfrac{1}{0} = \infty$

(3) Type-2 System : If $n = 2$; $G(s) = \dfrac{K'(s+z_1)(s+z_2)\cdots}{s^2(s+p_1)(s+p_2)\cdots}$

$\therefore$ (i) $e_{ss}(\text{position input}) = \dfrac{1}{1 + \underset{s\to 0}{Lt}\ G(s)H(s)}$

if $H(s) = 1$ $\therefore e_{ss} = \dfrac{1}{1 + \underset{s\to 0}{Lt}\ \dfrac{K'(s+z_1)(s+z_2)\cdots}{s^2(s+p_1)(s+p_2)\cdots}}$

$= \dfrac{1}{1+\infty} = \dfrac{1}{\infty} = 0$

(ii) $e_{ss}(\text{velocity}) = ?$

$e_{ss}(\text{velocity}) = \dfrac{1}{\underset{s\to 0}{Lt}\ sG(s)H(s)}$ and if $H(s) = 1$

$e_{ss}(\text{velocity}) = \dfrac{1}{\underset{s\to 0}{Lt}\ s\cdot \dfrac{K'(s+z_1)(s+z_2)\cdots}{s^2(s+p_1)(s+p_2)\cdots}} = \dfrac{1}{\infty} = 0$

(iii) $e_{ss}(\text{velocity}) = \dfrac{1}{\underset{s\to 0}{Lt}\ \cancel{s^2}\cdot \dfrac{K'(s+z_1)(s+z_2)\cdots}{\cancel{s^2}(s+p_1)(s+p_2)\cdots}} = \dfrac{1}{K'} = \dfrac{1}{K_a}$

Thus a type-2 system has zero position error, zero velocity error and a constant acceleration error.

Type of Input	Steady-state Error		
	Type-0 System	Type-1 System	Type-2 System
unit step input	$\frac{1}{1+K_p}$	0	0
unit-ramp	∞	$\frac{1}{K_v}$	0
unit-parabolic	∞	∞	$\frac{1}{K_a}$

Table: Steady-state Errors for various inputs and systems Types.

The error Constants K_p, K_v and K_a describe the ability of a system to reduce or eliminate steady-state errors. As the type of system becomes higher, progressively more errors (steady-state) are eliminated. In general, type-0, -1 and -2 are the most commonly employed systems in practice. Systems with type higher than 2 are not employed in practice because of two reasons.

(i) These are more difficult to stabilize

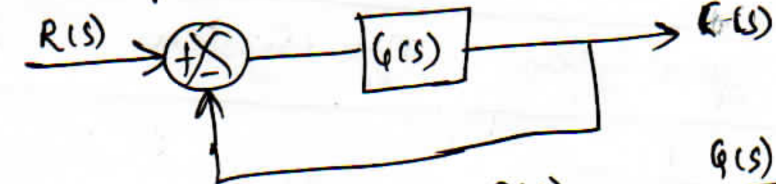
(ii) The dynamic errors for such systems tend to be larger than those for type-0, -1 and -2, although their steady state performance is desirable.

One of the disadvantages of error constants is that they do not give information on the steady-state error when inputs are other than the three basic types - step, ramp and parabolic. Another difficulty is that the error constants fail to indicate the exact manner in which error function change with time.

Problems

① obtain the response of unity feedback system whose open loop transfer function is $G(s) = \frac{4}{S(S+5)}$, when the input is unit step

(Sol)



The closed loop transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

$$\Rightarrow \frac{C(s)}{R(s)} = \frac{\frac{4}{S(S+5)}}{1+\frac{4}{S(S+5)}(1)} = \frac{4}{S(S+5)+4} = \frac{4}{S^2+5S+4} \rightarrow ①$$

given that $r(t) = u(t)$ $\therefore R(s) = \frac{1}{s} \longrightarrow ②$

from eqs ① & ②

$$C(s) = R(s)\frac{4}{S^2+5S+4} = \frac{4}{S(S^2+5S+4)} = \frac{4}{S(S+1)(S+4)}$$

The time response is obtained by taking inverse laplace transform of $C(s)$.

$$\therefore c(t) = L^{-1}C(s) = L^{-1}\left\{\frac{4}{S(S+1)(S+4)}\right\}$$

$$= L^{-1}\left\{\frac{1}{S} - \frac{4}{3}\cdot\frac{1}{S+1} + \frac{1}{3}\cdot\frac{1}{S+4}\right\}$$

$$= 1 - \frac{4}{3}e^{-t} + \frac{1}{3}e^{-4t}$$

② The response of a servo mechanism is $c(t) = 1+0.2e^{-60t} - 1.2e^{-10t}$ when subject to a unit step input. obtain an expression for closed loop transfer function. Determine undamped natural frequency and damping ratio.

(Sol) given that $c(t) = 1+0.2e^{-60t} - 1.2e^{-10t}$

taking LT on both sides

$$C(s) = 1 + 0.2\frac{1}{S+60} - 1.2\frac{1}{S+10}$$

$$= \frac{(S+60)(S+10) + 0.2(S+10) - 1.2(S+60)}{(S+60)(S+10)}$$

$$\Rightarrow C(s) = \frac{s^2+70s+600+0.2s^2+2s-1.2s^2-72s}{s(s+60)(s+10)} \quad (11)$$

$$= \frac{600}{s(s+10)(s+60)}$$

$\therefore$ The closed loop transfer function $= \frac{C(s)}{R(s)} = \frac{600}{(s+10)(s+60)}$

where $r(t) = u(t)$

$\therefore R(s) = 1/s$

$$\Rightarrow \frac{C(s)}{R(s)} = \frac{600}{s^2+70s+600} \longrightarrow (1)$$

The general form of 2nd order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \longrightarrow (2)$$

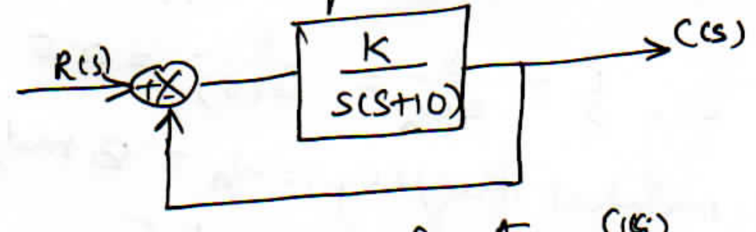
from eq (1) & (2) $\quad 2\zeta\omega_n = 70\ ; \quad \omega_n^2 = 600$

$\therefore$ undamped natural frequency $\omega_n = \sqrt{600}$ rad/sec

The damping factor $\zeta = \frac{70}{2\omega_n} = 1.43$

(3) A unity feedback system is characterized by an open loop transfer function $\frac{K}{s(s+10)}$. Determine the gain 'K' so that the system will have a damping ratio of 0.5. For this value of K determine peak over shoot and time at peak overshoot.

(Sol)



The closed loop transfer function $\frac{C(s)}{R(s)} = \frac{K}{s(s+10)+K}$

$$= \frac{K}{s^2+10s+K}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$2\zeta\omega_n = 10\ ; \quad \omega_n^2 = K \Rightarrow K = 100$

$2(0.5)\omega_n = 10$

$\Rightarrow \omega_n = 0$

$$\%M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100 = 16.3\%$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{10\sqrt{1-0.5^2}} = 0.363 \text{ sec}$$

(4) A closed loop servo is represented by the differential equation $\frac{d^2c}{dt^2} + 8\frac{dc}{dt} = 64e$. where c is the displacement of the output shaft and 'r' is the displacement of input shaft and $e = r - c$. Determine undamped natural frequency, damping ratio and percentage maximum overshoot for unit step input

(Sol) The system is represented by $\frac{d^2c}{dt^2} + 8\frac{dc}{dt} = 64e$

where $e = r - c$ r = input ; c = output

$$\therefore \frac{d^2c}{dt^2} + 8\frac{dc}{dt} = 64(r-c)$$

taking LT on both sides

$$s^2 C(s) + 8s\,C(s) = 64\left[R(s) - C(s)\right]$$

$$\text{or } C(s)\left[s^2+8s+64\right] = 64\,R(s)$$

$$\therefore TF = \frac{C(s)}{R(s)} = \frac{64}{s^2+8s+64} \longrightarrow ①$$

The general form of 2nd order system TF is $\longrightarrow ②$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

from eqs ① & ② $2\xi\omega_n = 8$; $\omega_n^2 = 64$

$\therefore \omega_n = 8$ rad/sec

$$\therefore \xi = \frac{8}{2\omega_n} = \frac{8}{2(8)} = 0.5$$

$\therefore$ undamped natural frequency $= \omega_n = 8$ rad/sec

damping factor $\xi = 0.5$

% Max peak overshoot $M_p = 100\, e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}$

$$= 100\, e^{\frac{-\pi \times 1/2}{\sqrt{1-(1/2)^2}}}$$

$$= 100\, e^{-\frac{\pi}{2\sqrt{3}/2}} = 100\, e^{-\pi/\sqrt{3}} = 16.37\%$$

(5) A system has the closed loop transfer function $\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$. It is required that the unit step response of the system should be have a settling time of 2 sec according to 2% criterion and the overshoot should be approximately 5%. What should be the closed loop pole locations.

(Sol) Given that $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

Settling time $t_s = \frac{4}{\xi\omega_n}$ for 2% tolerance

$= 2$ sec

$\therefore \xi\omega_n = \frac{4}{2} = 2 \longrightarrow (1)$

% peak overshoot $= 100\, e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} = 5$

$\therefore e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} = \frac{5}{100}$

taking 'ln' on both sides

$$\frac{-\pi\xi}{\sqrt{1-\xi^2}} = \ln(0.05) = -3$$

$$\therefore \frac{\pi^2\xi^2}{1-\xi^2} = 9 \Rightarrow \xi^2(\pi^2+9) = 9$$

$$\therefore \xi = \sqrt{\frac{9}{(9+\pi^2)}} = 0.69 \longrightarrow (2)$$

substituting eq (2) in (1) $\omega_n = \frac{2}{\xi} = 2.895$ rad/sec

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{8.39}{s^2+4s+8.39}$$

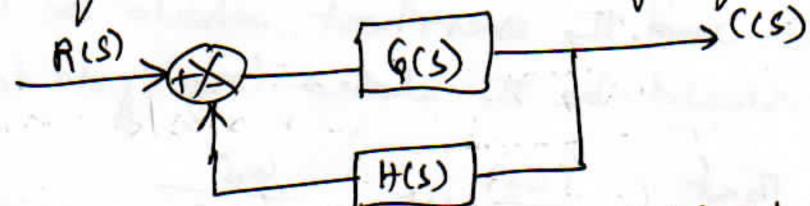
$$\therefore s = \frac{-4 \pm \sqrt{4^2-4(1)(8.39)}}{2} = \frac{-4 \pm j4.19}{2}$$

$$= -2 \pm j2.09$$

$\therefore$ The poles at $s_1 = -2+j2.09$ and $s_2 = -2-j2.09$

(6) For a unity feedback system, the open loop transfer function $G(s) = \frac{10}{s(s+2)}$; find the time domain specifications for a step input of 12 units.

(Sol)



Given that $G(s) = \frac{10}{s(s+2)}$; $H(s) = 1$

$\therefore$ The closed loop TF $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{10}{s(s+2)+10}$

$$\Rightarrow \frac{C(s)}{R(s)} = \frac{10}{s^2+2s+10} \longrightarrow (1)$$

The ... ral form of 2nd order system TF is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \longrightarrow (2)$$

Comparing eq (1) & (2)

$$\omega_n^2 = 10 \Longrightarrow \omega_n = \sqrt{10} \text{ rad/sec}$$

$$2\xi\omega_n = 2 \Longrightarrow \xi = \frac{2}{2\omega_n} = \frac{1}{\sqrt{10}}$$

The time domain specifications are

(i) Rise time $t_r = \frac{\pi - \phi}{\omega_d} = \frac{\pi - \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)}{\omega_n\sqrt{1-\xi^2}}$

$$= \frac{\pi - \tan^{-1}\left(\sqrt{1-1/10}\,/\,1/\sqrt{10}\right)}{\sqrt{10}\sqrt{9/10}} = \frac{\pi - \tan^{-1} 3}{3}$$

$= 0.63$ sec

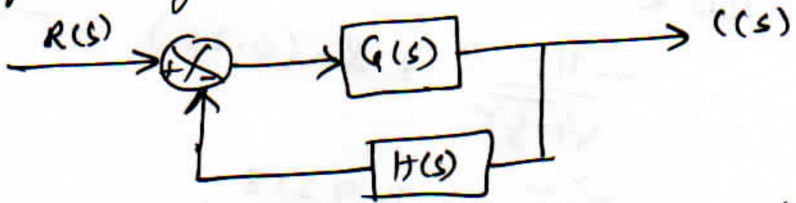
(ii) peak time $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{3} = 1.05$ sec

(iii) $\%M_p = 100\, e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} = 100\, e^{\frac{-\pi \cdot \frac{1}{\sqrt{10}}}{3/\sqrt{10}}} = 100\, e^{-\pi/3} = 35\%$.

(iv) $t_s = \frac{4}{\xi\omega_n} = \frac{4}{\frac{1}{\sqrt{10}}\sqrt{10}} = 4$ sec for 2% Tolerance

(1) The open loop transfer function of a unity feedback system is given by $G(s) = \frac{K}{S(TS+1)}$. where K and T are constants. By what factor should the amplifier gain be reduced so that the peak overshoot of unit ramp response of the system is reduced from 75% to 25%.

(Sol)



Given that $G(s) = \frac{K}{S(TS+1)}$; $H(S) = 1$

$\therefore$ The closed loop TF $= \frac{C(s)}{R(s)} = \frac{K}{S(TS+1)+K} = \frac{K}{S^2T+S+K}$

or $\frac{C(s)}{R(s)} = \frac{(K/T)}{s^2+\frac{1}{T}s+K/T} \longrightarrow (1)$

Also $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \longrightarrow (2)$

from eq (1) & (2) $\omega_n = \sqrt{K/T}$ and $2\zeta\omega_n = \frac{1}{T}$

$$\therefore \zeta = \frac{1}{2T\omega_n} = \frac{1}{2T\sqrt{K/T}}$$

$$= \frac{1}{2\sqrt{KT}} \longrightarrow (1)$$

For 75% peak over shoot let $K = K_1$

$$\therefore \%M_p = 100\, e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} = 75$$

$$e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} = 0.75$$

taking 'ln' on both sides

$$\frac{-\pi\zeta}{\sqrt{1-\zeta^2}} = \ln(0.75) = -0.2877$$

$$\frac{\pi^2\zeta^2}{1-\zeta^2} = 0.082$$

$$\Rightarrow \zeta^2(\pi^2+0.082) = 0.082$$

$$\therefore \zeta = \frac{\sqrt{0.082}}{\sqrt{\pi^2+0.082}} = \sqrt{\frac{0.082}{9.8764}} = 0.09 \longrightarrow (2)$$

from eqs ① & ②

$$\frac{1}{2\sqrt{KT}} = 0.09 \quad \therefore K_1 = \frac{1}{T}\cdot\frac{1}{4(0.09)^2}$$

$$= \frac{30.86}{T} \rightarrow \text{(i)}$$

for 25% peak over shoot let $K = K_2$

$$\therefore 100\, e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} = 25$$

$$\therefore \frac{-\pi\xi}{\sqrt{1-\xi^2}} = \ln(0.25) = -1.386$$

$$\frac{\pi^2\xi^2}{1-\xi^2} = 1.9218$$

$$\Rightarrow \xi^2(\pi^2 + 1.9218) = 1.9218$$

$$\text{and } \xi = \sqrt{\frac{1.9218}{\pi^2+1.9218}} = 0.4037 \rightarrow ③$$

from eqs ① & ③ $\quad \frac{1}{2\sqrt{K_2T}} = 0.4037$

$$\therefore K_2 = \frac{1}{T}\cdot\frac{1}{4(0.4037)^2} = 1.53/T \rightarrow \text{(ii)}$$

$$\therefore \frac{K_1}{K_2} = \frac{(30.86/T)}{(1.53/T)} = 20$$

∴ The gain should be reduced by a factor 20

⑥ The open loop transfer function of a unity feedback system is given by $G(s) = \frac{K}{s(s+1)(s+2)}$. Find the minimum value of 'K' for which the steady state error is less than 0.1 for unit ramp input

(Sol) Given that $G(s) = \frac{K}{s(s+1)(s+2)}$

$$H(s) = 1$$

$$r(t) = t \quad \therefore R(s) = \frac{1}{s^2}$$

The steady state error $e_{ss} = \underset{t\to\infty}{Lt}\, e(t) = \underset{s\to 0}{Lt}\, sE(s)$

$$= \underset{s\to 0}{Lt} \frac{sR(s)}{1+G(s)H(s)}$$

$$0.10 = \underset{s\to 0}{Lt} \frac{s\cdot\frac{1}{s^2}}{1+\frac{K}{s(s+1)(s+2)}}$$

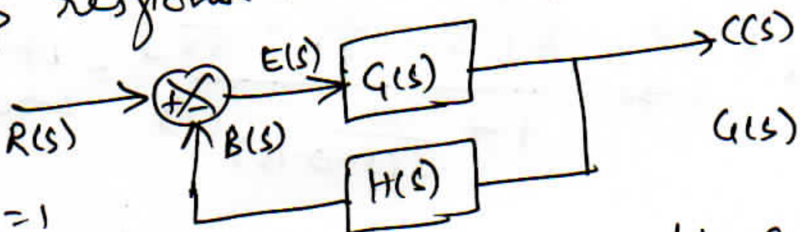
$$\Rightarrow 0.1 = \underset{s\to 0}{Lt} \frac{\left(\frac{1}{s}\right)}{\frac{s(s+1)(s+2)+K}{s(s+1)(s+2)}}$$

$$0.1 = \underset{s\to 0}{Lt} \frac{\left(\frac{1}{\cancel{s}}\right)\cancel{s}(s+1)(s+2)}{s(s+1)(s+2)+K} = \frac{2}{K}$$

$$\boxed{\therefore K = \frac{2}{0.1} = 20}$$

(7) A unity feedback control system has the closed loop transfer function $\frac{Ks+b}{s^2+as+b}$. Determine the steady state error in the unit ramp response in terms of K, a, and b.

(Sol)



$$G(s) = \frac{Ks+b}{s^2+as+b}$$

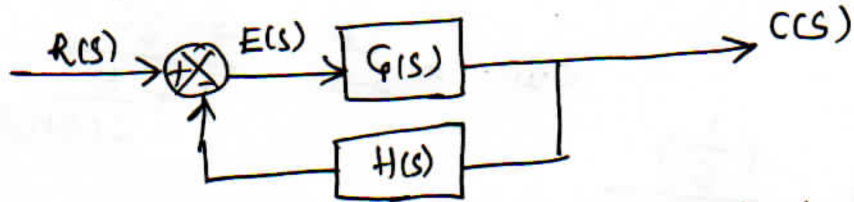
$r(t) = t$; $H(s) = 1$

Steady state error $e_{ss} = \underset{s\to 0}{Lt}\, sE(s) = \underset{s\to 0}{Lt} \frac{sR(s)}{1+G(s)H(s)}$

$$\Rightarrow e_{ss} = \underset{s\to 0}{Lt} \frac{s\cdot\frac{1}{s^2}}{1+\frac{(Ks+b)}{s^2+as+b}(1)} = \underset{s\to 0}{Lt} \frac{(1/s)(s^2+as+b)}{s^2+as+b+Ks+b} = \infty$$

① Find the steady state error as a function of time for the unity feedback system $G(S) = \frac{100}{S(1+0.1S)}$ for the input $r(t) = 1 + 2t + t^2/2$

(Sol)



Given that $G(s) = \frac{100}{s(1+0.1s)}$; $H(s) = 1$

Steady state Error $e_{ss} = \underset{t\to\infty}{Lt}\ e(t) = \underset{s\to 0}{Lt}\ sE(s)$

$$= \underset{s\to 0}{Lt}\ s\frac{R(s)}{1+G(s)H(s)}$$

where $R(s) = L[r(t)] = L[1+2t+t^2/2]$

$$= \frac{1}{s} + \frac{2}{s^2} + \frac{1}{s^3}$$

$$\therefore e_{ss} = \underset{s\to 0}{Lt}\ \frac{s\left[\frac{1}{s}+\frac{2}{s^2}+\frac{1}{s^3}\right]}{1+\frac{100}{s(\ +0.1s)}(1)} = \underset{s\to 0}{Lt}\ \frac{s\left[\frac{1}{s}+\frac{2}{s^2}+\frac{1}{s^3}\right]}{\frac{s(\ +0.1s)+100}{s(1+0.1s)}}$$

$$= \underset{s\to 0}{Lt}\ \frac{s\cdot\frac{1}{s}\,s(1+0.1s)}{s(1+0.1s)+100} + \underset{s\to 0}{Lt}\ \frac{s\cdot\frac{2}{s^2}\,s(0.1s+1)}{s(0.1s+1)+00} +$$

$$\underset{s\to 0}{Lt}\ \frac{s\cdot\frac{1}{s^3}\,s(0.1s+1)}{s(0.1s+1)+100}$$

$$e_{ss} = \underset{s\to 0}{Lt}\ \frac{s(1+0.1s)}{s(0.1s+1)+100} + \underset{s\to 0}{Lt}\ \frac{2(0.1s+1)}{s(0.1s+1)+00} + \underset{s\to 0}{Lt}\ \frac{(0.1s+1)}{s[s(0.1s+1)+00]}$$

$$= 0 + \frac{2}{100} + \frac{1}{0}$$

$$= 0 + \frac{2}{100} + \infty = \infty$$