

# Transmission Lines – Basic Theories

## (Intensive Reading)

### 1 Introduction

In previous sections on plane waves, we have developed an electromagnetic model with which we can analyse electromagnetic actions that occur at a distance and which are caused by time-varying charges and currents. These actions were explained in terms of waves and electromagnetic fields.

For efficient point-to-point transmission of power and information, the source energy must be directed or guided. We will study transverse electromagnetic (TEM) waves guided by transmission lines. TEM waves are characterised by the fact that  $\mathbf{E}$  and  $\mathbf{H}$  are perpendicular to each other and both are transverse to the direction of propagation along the guiding structure. Associated with the  $\mathbf{E}$  and  $\mathbf{H}$  fields of transmission lines, we can define voltages and currents  $V$  and  $I$  flowing on the transmission line. Plane waves are unguided TEM waves, and we will see that there are many similarities between  $\mathbf{E}$  and  $\mathbf{H}$  of plane waves, and  $V$  and  $I$  of transmission lines. For a wave to be purely TEM, it has to propagate in regions of homogeneous dielectric properties.

### 2 Common types of transmission lines

Common types of transmission lines are shown in Figure 1, and include:

- *Parallel plate transmission line*: This type of transmission line consists of two parallel conducting plates separated by a dielectric slab (or air) of uniform thickness. The plates should theoretically be infinitely wide to support TEM waves. If a finite width is used, this may become impractical, since leakage may occur from the sides of the parallel plate guide. Sometimes a thin conducting strip is placed in the centre of the dielectric slab, and this then becomes *stripline* (see Figure 2b). Stripline is of more practical use, since the electromagnetic energy may effectively be trapped in a region surrounding the strip.
- *Two-wire transmission line*: This transmission line consists of a pair of two parallel conducting wires separated by a uniform distance. Examples are overhead power and telephone lines.
- *Coaxial transmission line*: This consists of an inner conductor and a coaxial outer sheath separated by a dielectric medium. This structure has the important advantage of confining the electric and magnetic fields entirely within a homogeneous dielectric medium. No stray fields are generated by a coaxial transmission line, and very little external interference is coupled into the transmission line. This is often used as TV cables, and also as input cables to high-frequency precision measuring instruments.
- *Microstrip lines*: Microstrip transmission line is a kind of high-grade printed circuit construction, consisting of a track of copper or gold on an insulating substrate. There is a ground plane on the other side of the insulating substrate, formed from similar conductor. Fields propagate in both the dielectric and air, and therefore the fields are not purely TEM in nature. However, for a high dielectric constant, most of the energy is confined to the dielectric, and the fields are said to be quasi-TEM (“nearly” TEM) in nature, so that we may use TEM approximations in characterising the fields and waves on the microstrip line. It is one of the most popular planar transmission lines, because it can easily be manufactured by photolithographic techniques and is easily integrated with other passive and active microwave devices.

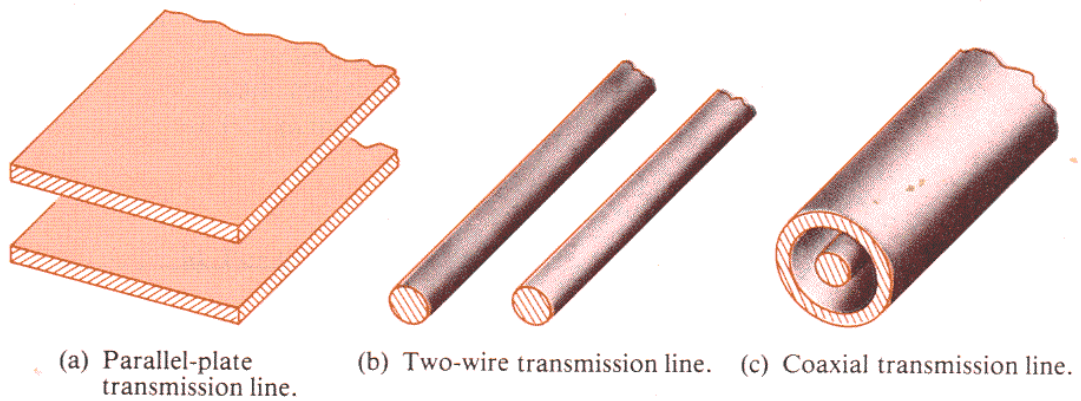


Figure 1: Common types of transmission lines

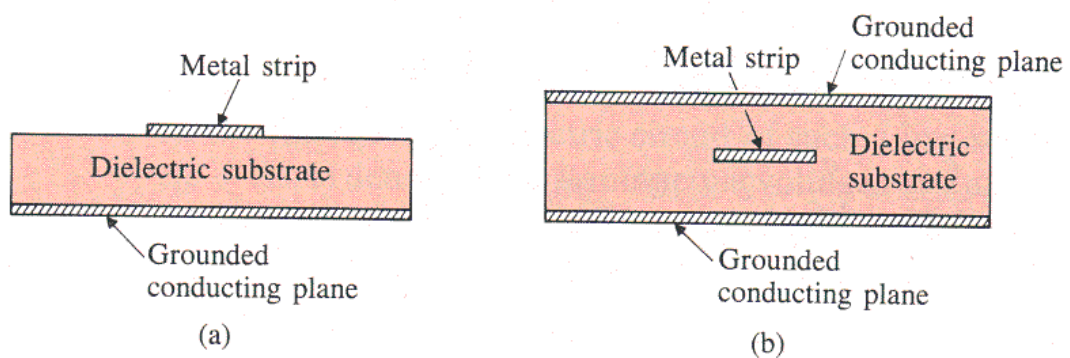


Figure 2: (a) Microstrip and (b) stripline.

### 3 Transmission line characteristics

A transmission line is characterised by:

- The characteristic impedance  $Z_0$ , which is the ratio of voltage to current of a wave travelling in one direction. If the transmission line is lossless,  $Z_0$  is real-valued, and no power is dissipated as a wave travels along the transmission line. A lossy transmission line has a complex characteristic impedance, and some of the power travelling down the transmission line is converted into heat which cannot be recovered.

The characteristic impedance of a transmission line is a function of its dimensions and the properties of the dielectric material used in the construction of the transmission line.

- The wavenumber  $k = \beta - j\alpha$  of the transmission line, where  $\beta$  and  $\alpha$  are the phase constant and attenuation constant of the transmission line. For a lossless transmission line,  $k = \beta$  is a real number, while for lossy transmission lines,  $\alpha$  is non-zero so that  $k$  becomes complex.<sup>1</sup>

If the conductors are assumed to be ideal, the phase constant and attenuation constant is purely a function of the properties of the dielectric material used in the

<sup>1</sup> Sometimes the propagation constant  $\gamma = \alpha + j\beta$  is used instead of the wavenumber  $k$ . The relation between  $\gamma$  and  $k$  is thus merely  $\gamma = jk$  or  $k = -j\gamma$ .

construction of the transmission line. In practice, the dielectrics have a relative permeability of  $\mu_r = 1$  and a relative permittivity of  $\epsilon_r = \epsilon_r' - j\epsilon_r''$ . We may then calculate  $k$  in exactly the same way as we did for plane waves, i.e.

$$\begin{aligned} k &= \omega\sqrt{\mu\epsilon} = \omega\sqrt{\mu_0\epsilon_0}\sqrt{\mu_r\epsilon_r} \\ &= k_0\sqrt{\epsilon_r' - j\epsilon_r''} \\ &= \beta - j\alpha. \end{aligned} \quad (1)$$

The wavelength and phase velocity may then also be calculated as for plane waves:

$$\lambda = \frac{2\pi}{\beta}, \quad (2)$$

while the phase velocity is given by

$$u_p = \frac{1}{\text{Re}[\sqrt{\mu\epsilon}]}. \quad (3)$$

Manufacturers of coaxial transmission lines often specify the characteristic impedance, the loss in dB/m and the phase velocity as a fraction of the speed of light,  $u_p / c$ . The attenuation constant and phase constant are then obtained from

$$\text{Loss in dB/m} = 20 \log(e^\alpha) = 8.686 \alpha \quad (4)$$

and

$$\beta = \omega / u_p. \quad (5)$$

Instead of specifying the characteristic impedance, phase constant and attenuation constant, transmission lines may also be characterised by their inductance  $L$ , capacitance  $C$ , resistance  $R$  and conductance  $G$  per unit length. Note that the resistance  $R$  is related to the ohmic losses in the conductors, whereas the conductance  $G$  is determined by losses in the dielectrics. In general, the following relations between the distributed parameters and the material properties of the dielectric hold<sup>2</sup>:

$$\begin{aligned} LC &= \mu\epsilon \\ G/C &= \sigma/\epsilon \end{aligned} \quad (6)$$

A section of a transmission line is then modelled using the equivalent circuit shown in Figure 3.

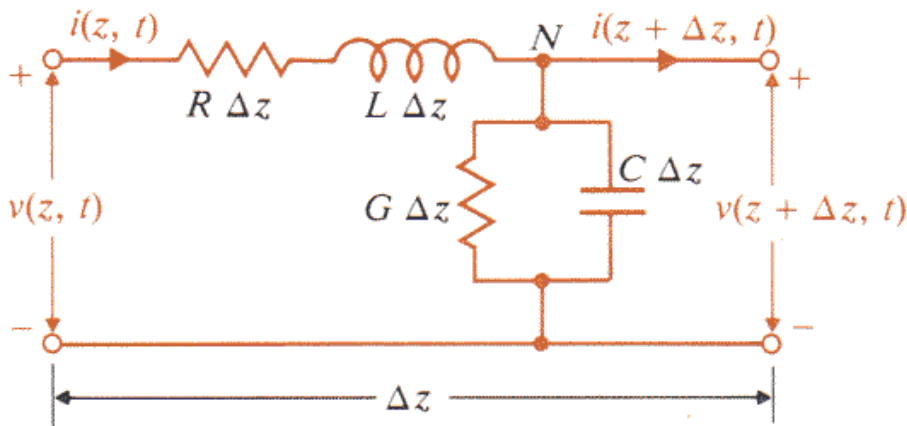


Figure 3: Equivalent circuit of a differential length  $\Delta z$  of a transmission line.

<sup>2</sup> The term  $\epsilon$  in (6) actually refers to the real part of the complex permittivity of the lossy dielectric. Strictly speaking, the relations are  $LC = \mu\epsilon'$ ,  $G/C = \sigma/\epsilon'$ .

If these quantities are known, the characteristic impedance, phase constant and attenuation constant and phase velocity is obtained from

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \quad (7)$$

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} \quad (8)$$

$$u_p = \frac{\omega}{\beta}. \quad (9)$$

For *lossless transmission lines*,  $R = G = 0$ , so that (7) and (8) reduce to

$$Z_0 = \sqrt{\frac{L}{C}} \quad (10)$$

$$\beta = \omega\sqrt{LC} \quad (11)$$

$$\alpha = 0. \quad (12)$$

Note that the distributed parameters may be dependent on frequency, especially the resistance  $R$ . We then find that  $\beta$  is not a linear function of frequency, and therefore from (9), we also find that the phase velocity is not constant for all frequencies. In such cases, high-frequency components of signals will travel at a different velocity compared to low-frequency components, causing signal distortion. Such transmission lines are called *dispersive*. However, if a materials from which a transmission line are made are lossy, it may still be possible to manipulate them to so that  $R/L = G/C$ . The transmission line is then called *distortionless*. Even though it is still lossy ( $\alpha \neq 0$ ),  $\beta$  becomes a linear function of frequency, while  $u_p$  is constant for all frequencies.  $\beta$  and  $Z_0$  are then given by (11) and (10), while

$$\alpha = R\sqrt{C/L}. \quad (13)$$

Table 1 shows the distributed parameters of a two-wire and co-axial transmission lines.

Parameter	Two-wire line	Coaxial line
$R$ ( $\Omega/\text{m}$ )	$\frac{R_s}{\pi a}$	$\frac{R_s}{2\pi} \left( \frac{1}{a} + \frac{1}{b} \right)$
$L$ (H/m)	$\frac{\mu}{\pi} \cosh^{-1}(D/2a)$	$\frac{\mu}{2\pi} \ln(b/a)$
$G$ (S/m)	$\frac{\pi\sigma}{\cosh^{-1}(D/2a)}$	$\frac{2\pi\sigma}{\ln(b/a)}$
$C$ (F/m)	$\frac{\pi\epsilon}{\cosh^{-1}(D/2a)}$	$\frac{2\pi\epsilon}{\ln(b/a)}$

Table 1: Distributed parameters of a two-wire transmission line with wire radius  $a$  and separated by a distance  $D$ , and a coaxial transmission line with a centre conductor radius of  $a$  and outer conductor radius  $b$ .  $\sigma = \omega\epsilon''$  and  $\mu$  is the equivalent conductivity and permeability of the dielectric.

The surface resistivity is given by

$$R_s = \sqrt{\pi f \mu_c / \sigma_c}, \quad (14)$$

while  $\sigma_c$  and  $\mu_c$  are parameters pertaining to the conductor.

In the case of microstrip, the wave does not propagate in a homogeneous dielectric. For this reason, the microstrip line does not support a pure TEM wave, since the phase velocity of TEM fields would be  $c/\sqrt{\epsilon_r}$ , but the phase velocity in the air would be  $c$ . A phase match at the dielectric interface would then be impossible. For the quasi-TEM fields, we find that

$$u_p = \frac{c}{\sqrt{\epsilon_{\text{eff}}}} \quad (15)$$

and

$$\beta = \omega\sqrt{\mu_0\epsilon_0\epsilon_{\text{eff}}} = k_0\sqrt{\epsilon_{\text{eff}}} \quad (16)$$

where  $\epsilon_{\text{eff}}$  is the effective dielectric constant of the microstrip line. In general,  $1 < \epsilon_{\text{eff}} < \epsilon_r$ , and it is dependent on the substrate thickness,  $d$ , and the conductor width  $W$ . If  $\epsilon_r$  is the real part of the relative permittivity of the dielectric, the effective dielectric constant of a microstrip line is given approximately by

$$\epsilon_{\text{eff}} = \frac{\epsilon_r + 1}{2} + \frac{\epsilon_r - 1}{2} \frac{1}{\sqrt{1 + 12d/W}} \quad (17)$$

Given the dimensions of the microstrip, the characteristic impedance can be calculated as

$$Z_0 = \begin{cases} \frac{60}{\sqrt{\epsilon_{\text{eff}}}} \ln\left(\frac{8d}{W} + \frac{W}{4d}\right) & W/d < 1 \\ \frac{120\pi}{\sqrt{\epsilon_{\text{eff}}}[W/d + 1.393 + 0.667 \ln(W/d + 1.444)]} & W/d > 1 \end{cases} \quad (18)$$

Alternatively, given the characteristic impedance  $Z_0$ , we can calculate the appropriate line width from

$$W/d = \begin{cases} \frac{8e^A}{e^{2A} - 2} & W/d < 2 \\ \frac{2}{\pi} \left[ B - 1 - \ln(2B - 1) + \frac{\epsilon_r - 1}{2\epsilon_r} \left( \ln(B - 1) + 0.39 - \frac{0.61}{\epsilon_r} \right) \right] & W/d > 2 \end{cases} \quad (19)$$

where  $A = \frac{Z_0}{60} \sqrt{\frac{\epsilon_r + 1}{2}} + \frac{\epsilon_r - 1}{\epsilon_r + 1} \left( 0.23 + \frac{0.11}{\epsilon_r} \right)$  and  $B = \frac{377\pi}{2Z_0\sqrt{\epsilon_r}}$ .

With a dielectric loss tangent of  $\tan\delta$ , the attenuation constant is given by the sum of the contributions from dielectric losses and conductor losses:

$$\alpha = \alpha_d + \alpha_c = \frac{k_0\epsilon_r(\epsilon_{\text{eff}} - 1)\tan\delta}{2\sqrt{\epsilon_{\text{eff}}}(\epsilon_r - 1)} + \frac{R_s}{Z_0W} \quad (20)$$

with  $R_s$  given by (14).

#### 4 Solutions to the transmission line equation

The coupled time-harmonic transmission line equations for the phasor voltage and current on a transmission line aligned along the  $z$ -axis are

$$\frac{d^2V(z)}{dz^2} + k^2V(z) = 0 \quad (21)$$

and

$$\frac{d^2 I(z)}{dz^2} + k^2 I(z) = 0. \quad (22)$$

The solutions of (21) and (22) are

$$V(z) = V^+(z) + V^-(z) = V_0^+ e^{-jkz} + V_0^- e^{+jkz}, \quad (23)$$

and

$$I(z) = I^+(z) + I^-(z) = I_0^+ e^{-jkz} + I_0^- e^{+jkz}, \quad (24)$$

where the plus and minus signs denote waves travelling in the  $+z$  and the  $-z$  directions. The wave amplitudes are related via

$$\frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = Z_0. \quad (25)$$

For waves travelling on an infinitely long transmission line, there can be only forward travelling waves in the  $+z$  direction, and the second terms on the right sides of (23) and (24), representing the reflected waves, vanish. The same is also true for finite lines terminated in the characteristic impedance of the transmission line. We say that such a line is matched.

Consider a general case of a finite transmission line of length  $l$ , characteristic impedance  $Z_0$  and terminated in an arbitrary impedance  $Z_L$ , as depicted in Figure 4. A sinusoidal voltage source  $V_g$  with internal impedance  $Z_g$  is connected to the line at  $z = -d$ . In such case,

$$\frac{V(z=0)}{I(z=0)} = \frac{V_L}{I_L} = Z_L \quad (26)$$

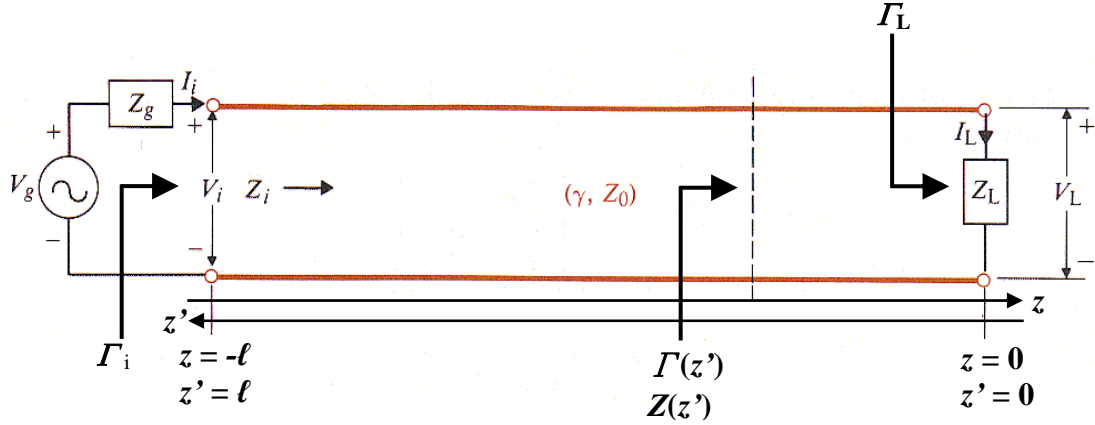


Figure 4: Finite transmission line terminated in load impedance  $Z_L$ .

Obviously, a forward-travelling wave alone would not be able to satisfy (26). Therefore a reflected wave will exist on unmatched transmission lines.

Given  $Z_0$ ,  $Z_L$ ,  $k$  and  $z'$ , we may substitute (25) into (24) and then evaluate  $V(z=0)$  and  $I(z=0)$ :

$$\begin{aligned} V_0^+ + V_0^- &= V_L \\ \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0} &= I_L \end{aligned} \quad (27)$$

From (27), we find that  $V_0^+ = \frac{1}{2} I_L (Z_L + Z_0)$  and  $V_0^- = \frac{1}{2} I_L (Z_L - Z_0)$ . Substituting these into (23) and (24) and setting  $z' = -z$  yields

$$\begin{aligned}
V &= V^+ + V^- = \frac{I_L(Z_L + Z_0)}{2} e^{jkz'} + \frac{I_L(Z_L - Z_0)}{2} e^{-jkz'} \\
I &= I^+ + I^- = \frac{I_L(Z_L + Z_0)}{2Z_0} e^{jkz'} - \frac{I_L(Z_L - Z_0)}{2Z_0} e^{-jkz'} .
\end{aligned} \tag{28}$$

The expressions in (28) can also be rewritten as

$$\begin{aligned}
V &= I_L \left[ Z_L \frac{e^{jkz'} + e^{-jkz'}}{2} + Z_0 \frac{e^{jkz'} - e^{-jkz'}}{2} \right] = I_L [Z_L \cos(kz') + jZ_0 \sin(kz')] \\
I &= \frac{I_L}{Z_0} \left[ Z_0 \frac{e^{jkz'} + e^{-jkz'}}{2} + Z_L \frac{e^{jkz'} - e^{-jkz'}}{2} \right] = \frac{I_L}{Z_0} [Z_0 \cos(kz') + jZ_L \sin(kz')] .
\end{aligned} \tag{29}$$

In (29), the complex trigonometric functions can be calculated from  $\cos kz' = \cos(\beta z' - j\alpha z') = \cos \beta z' \cosh \alpha z' + j \sin \beta z' \sinh \alpha z'$  and  $\sin kz' = \sin(\beta z' - j\alpha z') = \sin \beta z' \cosh \alpha z' - j \cos \beta z' \sinh \alpha z'$ .

Using (29), the impedance seen looking into the transmission line towards the load at a distance  $z'$  from the load can be written as

$$\begin{aligned}
Z(z') &= \frac{V}{I} = Z_0 \frac{Z_L \cos(kz') + jZ_0 \sin(kz')}{Z_0 \cos(kz') + jZ_L \sin(kz')} \\
&= Z_0 \frac{Z_L + jZ_0 \tan(kz')}{Z_0 + jZ_L \tan(kz')} .
\end{aligned} \tag{30}$$

At the generator,  $z' = l$ , so that the input impedance  $Z_i$  seen by the generator becomes

$$Z_i = Z(z' = l) = Z_0 \frac{Z_L + jZ_0 \tan(kl)}{Z_0 + jZ_L \tan(kl)} . \tag{31}$$

Equation (31) can be used to evaluate the input impedance of a lossy transmission line. As far as generator is concerned, the terminated transmission line can be replaced by  $Z_i$  as shown in Figure 5.

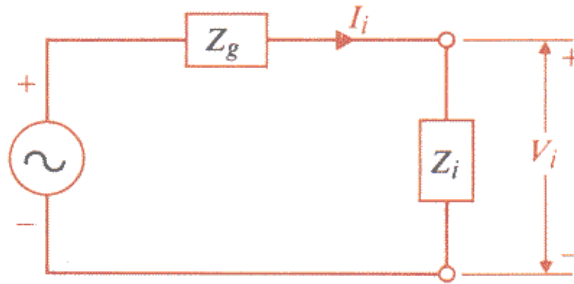


Figure 5: Equivalent circuit for terminated transmission line at the generator end.

The ratio of the complex amplitudes of the reflected and the incident voltage waves at the load ( $z' = 0$ ) is called the voltage reflection coefficient. Using (28), it can be calculated as

$$\Gamma_L = \frac{V^-}{V^+} \Big|_{z'=0} = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma_L| e^{j\theta} . \tag{32}$$

At a distance  $z'$  from the load, the input impedance seen looking into the transmission line toward the load can also be calculated by using (28) instead of (29), giving

$$\begin{aligned}
Z(z') &= \frac{V}{I} = Z_0 \frac{1 + \frac{Z_L - Z_0}{Z_L + Z_0} e^{-j2kz'}}{1 - \frac{Z_L - Z_0}{Z_L + Z_0} e^{-j2kz'}} \\
&= Z_0 \frac{1 + \Gamma_L e^{-j2kz'}}{1 - \Gamma_L e^{-j2kz'}}.
\end{aligned} \tag{33}$$

We can rewrite (33) as

$$Z(z') = Z_0 \frac{1 + \Gamma(z')}{1 - \Gamma(z')}, \tag{34}$$

which implies that

$$\Gamma(z') = \frac{Z(z') - Z_0}{Z(z') + Z_0} = \Gamma_L e^{-j2kz'}, \tag{35}$$

where  $\Gamma(z')$  is the equivalent reflection coefficient as seen at a distance  $z'$  from the load. Note that the same result can be obtained by calculating the equivalent reflection coefficient as  $\Gamma(z') = V^- / V^+$ , with  $V^-$  and  $V^+$  defined in (28).

Our discussion has thus far been restricted to the effects of the load on the input impedance and on the characteristics of voltage and current waves. No attention has been paid to the generator at the other end of the transmission line. Consider the case where we connect a voltage source  $V_g$  with internal impedance  $Z_g$  to a transmission line of length  $l$  and terminated in a load impedance  $Z_L$ , as shown in Figure 4. In addition to the reflection at the load, we find that when  $Z_g \neq Z_0$ , waves experience a reflection at the source end as well. The voltage and current on the transmission line are then a superposition of multiple reflections. The voltage reflection coefficient at the generator end is given by

$$\Gamma_g = \frac{Z_g - Z_0}{Z_g + Z_0}. \tag{36}$$

After some mathematical manipulation, we may express the voltage and current at any position along the transmission line in terms of the source characteristics ( $V_g, Z_g$ ) as

$$V = \frac{Z_0 V_g e^{-jk(l-z')}}{Z_0 + Z_g} \left( \frac{1 + \Gamma_L e^{-j2kz'}}{1 - \Gamma_g \Gamma_L e^{-j2kl}} \right) \tag{37}$$

and

$$I = \frac{V_g e^{-jk(l-z')}}{Z_0 + Z_g} \left( \frac{1 - \Gamma_L e^{-j2kz'}}{1 - \Gamma_g \Gamma_L e^{-j2kl}} \right). \tag{38}$$

**Example 1:** A generator with an open-circuit voltage of  $v_g(t) = 10 \sin(8000\pi t)$  V and an internal impedance of  $Z_g = 40 + j30 \Omega$  is connected to a  $50 \Omega$  distortionless transmission line. The line has a resistance of  $0.5 \Omega/\text{m}$ , and its lossy dielectric medium has a loss tangent of  $0.18\%$ . The line is  $50 \text{ m}$  long and is terminated in a matched load. Find (a) the instantaneous expressions for the voltage and current at an arbitrary position on the line, (b) the instantaneous expression for the voltage and the current at the load, and (c) the average power transmitted to the load.



**Solution:** Given  $R = 0.5 \text{ } \Omega/\text{m}$ ,  $\tan \delta = 0.0018$ ,  $l = 50 \text{ m}$ ,  $Z_g = 40 + j30 \text{ } \Omega$ ,  $Z_0 = Z_L = 50 \text{ } \Omega$

$$\tan \delta = \frac{\sigma}{\omega \epsilon} = \frac{1}{\omega} \frac{\sigma}{\epsilon} = \frac{1}{\omega} \frac{G}{C}$$

Therefore  $\frac{G}{C} = \omega \tan \delta = (8000\pi)(0.0018) = 45.2$

For a distortionless line we know that  $\frac{R}{L} = \frac{G}{C}$ .

Thus  $\frac{R}{L} = 45.2$  and  $L = R/45.2 = 0.011 \text{ H/m}$ .

For a distortionless line, we also know that  $Z_0 = \sqrt{\frac{L}{C}}$ . Therefore  $C = \frac{L}{Z_0^2} = 4.42 \text{ } \mu\text{F/m}$ .

Furthermore,  $\alpha = R\sqrt{C/L} = R/Z_0 = 0.01 \text{ Np/m}$  and  $\beta = \omega\sqrt{LC} = 5.55 \text{ rad/m}$ .

With a cosine reference,  $V_g = -j10 = 10 e^{-j\pi/2}$ . The reflection coefficient at the load is given by

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = 0.$$

The phasor voltage and current on the transmission line obtained from (37) and (38) with  $z = l - z'$ :

$$\begin{aligned} V(z) &= \frac{Z_0 V_g e^{-jkz}}{Z_0 + Z_g} \left( \frac{1 + \Gamma_L e^{-j2k(l-z)}}{1 - \Gamma_g \Gamma_L e^{-j2kl}} \right) \\ &= \frac{Z_0 V_g e^{-\alpha z} e^{-j\beta z}}{Z_0 + Z_g} \cdot 1 \\ &= \frac{(50)(-j10) e^{-0.01z} e^{-j5.55z}}{50 + 40 + j30} = 5.27 e^{-j1.8925} e^{-0.01z} e^{-j5.55z} \text{ V} \end{aligned}$$

$$\begin{aligned} I(z) &= \frac{V_g e^{-jkz}}{Z_0 + Z_g} \left( \frac{1 - \Gamma_L e^{-j2k(l-z)}}{1 - \Gamma_g \Gamma_L e^{-j2kl}} \right) \\ &= \frac{V_g e^{-jkz}}{Z_0 + Z_g} = V(z) / Z_0 = 0.1054 e^{-j1.8925} e^{-0.01z} e^{-j5.55z} \text{ A} \end{aligned}$$

Instantaneous voltage and current on the transmission line:

$$v(z, t) = \text{Re} \left[ V(z) e^{j\omega t} \right] = 5.27 e^{-0.01z} \cos(8000\pi t - 5.55z - 1.8925) \text{ V}$$

$$i(z, t) = \text{Re} \left[ I(z) e^{j\omega t} \right] = 0.1054 e^{-0.01z} \cos(8000\pi t - 5.55z - 1.8925) \text{ A}$$

(b) Phasor load voltage and current:

$$V_L = V(50) = 3.1964 e^{-j279.3925} = 3.1964 e^{-j2.9323} \text{ V}$$

$$I_L = I(50) = 0.0639 e^{-j279.3925} = 0.0639 e^{-j2.9323} \text{ A}$$

Instantaneous load voltage and current:

$$v_L = v(50, t) = \text{Re} \left[ V(50) e^{j\omega t} \right] = 3.1964 \cos(8000\pi t - 2.9323) \quad \text{V}$$

$$i_L = i(50, t) = \text{Re} \left[ I(50) e^{j\omega t} \right] = 0.0639 \cos(8000\pi t - 2.9323) \quad \text{A}$$

(c) Average power delivered to the load:

$$P_{\text{av},L} = \frac{1}{2} \text{Re} \left[ V_L I_L^* \right] = 0.1022 \quad \text{W}$$

## 5 Waves on lossless transmission lines

Corresponding expressions for the voltage, current, impedance and reflection coefficient on a lossless transmission line can easily be obtained from the corresponding equations for lossy transmission lines by setting  $k = \beta$ . Thus

$$\begin{aligned} V &= \frac{I_L(Z_L + Z_0)}{2} e^{j\beta z'} + \frac{I_L(Z_L - Z_0)}{2} e^{-j\beta z'} = I_L [Z_L \cos(\beta z') + jZ_0 \sin(\beta z')] \\ I &= \frac{I_L(Z_L + Z_0)}{2Z_0} e^{j\beta z'} - \frac{I_L(Z_L - Z_0)}{2Z_0} e^{-j\beta z'} = \frac{I_L}{Z_0} [Z_0 \cos(\beta z') + jZ_L \sin(\beta z')], \end{aligned} \quad (39)$$

and

$$Z(z') = Z_0 \frac{Z_L + jZ_0 \tan(\beta z')}{Z_0 + jZ_L \tan(\beta z')}, \quad (40)$$

or

$$Z(z') = Z_0 \frac{1 + \Gamma(z')}{1 - \Gamma(z')}, \quad (41)$$

where

$$\Gamma(z') = \Gamma_L e^{-j2\beta z'} = |\Gamma_L| e^{j(\theta - 2\beta z')} \quad (42)$$

and  $\Gamma_L = |\Gamma_L| e^{j\theta}$  is defined by (32). Expressions for the voltage and current in the case of a mismatch at the source can similarly be found by substituting  $k = \beta$  in (37) and (38).

The voltage on the transmission line consists of the sum of a travelling wave and a standing wave. The magnitude of the voltage therefore displays maxima and minima at fixed positions, as depicted in Figure 6.

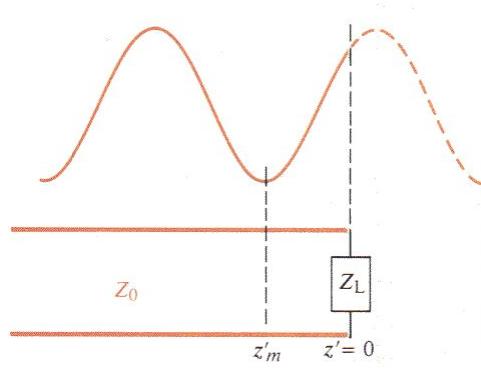


Figure 6: Magnitude of voltage for a transmission line terminated in a complex load impedance.

From (28), we rewrite the voltage on the transmission line as

$$\begin{aligned}
V &= \frac{I_L(Z_L + Z_0)}{2} e^{j\beta z'} \left[ 1 + \frac{Z_L - Z_0}{Z_L + Z_0} e^{-j2\beta z'} \right] \\
&= V_0^+ e^{-j\beta(l-z')} (1 + \Gamma_L e^{-j2\beta z'}) \\
&= V_0^+ e^{-j\beta z} (1 + |\Gamma_L| e^{j(\theta-2\beta z')}).
\end{aligned}$$

Similarly,

$$I = \frac{V_0^+}{Z_0} e^{-j\beta z} (1 - |\Gamma_L| e^{j(\theta-2\beta z')}).$$

$|V| = V_{\max}$  at points where  $e^{j(\theta-2\beta z')} = +1$ , and  $|V| = V_{\min}$  where  $e^{j(\theta-2\beta z')} = -1$ . We may define the ratio of the maximum to the minimum voltage along a finite, terminated line as the standing-wave ratio (SWR) or voltage standing-wave ratio (VSWR):

$$S = \frac{V_{\max}}{V_{\min}} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}. \quad (43)$$

$S$  is a dimensionless quantity that ranges from 1 to  $\infty$ . On a lossless transmission line ( $k = \beta$ ,  $Z_0$  real valued), we have the following special cases indicated in Table 2.

$\Gamma_L$	$S$	Condition
0	1	Matched load $Z_L = Z_0$
-1	$\infty$	Short circuit $Z_L = 0$
1	$\infty$	Open circuit $Z_L = \infty$

Table 2: Special cases of a matched load, short circuit and open circuit with corresponding reflection coefficient and standing-wave ratio.

From (41) and (42), we see that the input impedance becomes real and equal to  $S Z_0$  when  $e^{j(\theta-2\beta z')} = +1$ . At these points, we have a voltage maximum and a current minimum. On the other hand, at those points where  $e^{j(\theta-2\beta z')} = -1$ , we have voltage minima and current maxima. We can summarise it as follows:

Position	Input impedance	Condition
$z'_M = \frac{\theta + 2n\pi}{2\beta} \quad n = 0, 1, 2, \dots$	$Z(z'_M) = S Z_0$	$ V _{\max}$ , $ I _{\min}$
$z'_m = \frac{\theta + (2n+1)\pi}{2\beta} \quad n = 0, 1, 2, \dots$	$Z(z'_m) = Z_0 / S$	$ V _{\min}$ , $ I _{\max}$

Table 3: Positions of voltage and current maxima and minima on a terminated transmission line.

Refer to Figures 7 and 8 for the cases where the transmission line is terminated in a short circuit, open circuit and resistive load.

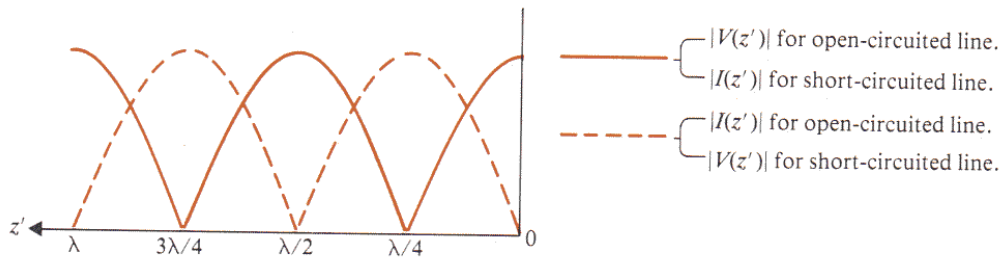


Figure 7: Voltage and current standing waves on open- and short-circuited lossless lines.

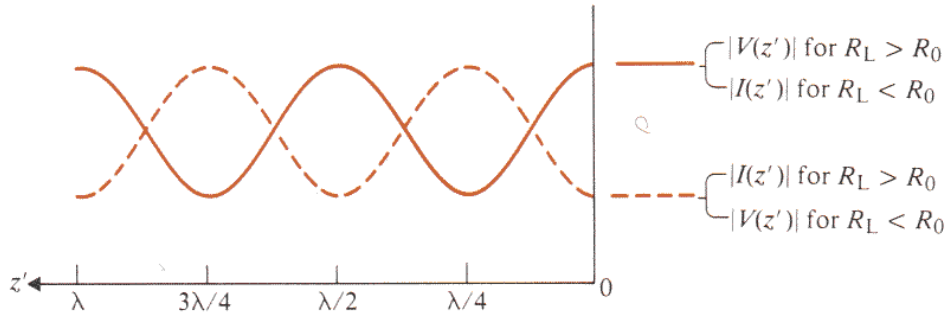


Figure 8: Voltage and current standing waves on a lossless transmission line with characteristic impedance  $Z_0 = R_0$  and terminated in a resistive load  $Z_L = R_L$ .

Note that the distance between successive voltage maxima is  $\lambda/2$ , and not  $\lambda$ . This is due to the fact that we are considering the amplitude of the voltage, and not the maximum voltage *per se*.

## 6 Transmission lines as circuit elements

Transmission lines are not only used as wave-guiding structures, but they may also serve as circuit elements. We consider a few examples here.

### (i) Lossless transmission line with an open circuit termination

The input impedance of a lossless transmission line of length  $l$  is given by (40) where  $Z_0$  is real valued:

$$Z_i = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)} \quad (44)$$

For an open circuit,  $Z_L = \infty$ , and then

$$Z_{i,o} = jX_{i,o} = -jZ_0 \cot \beta l = -jZ_0 \cot(2\pi l / \lambda) \quad (45)$$

Equation (45) shows that the input impedance of an open-circuited lossless transmission line is purely reactive. The line can be either capacitive or inductive, depending on the value of  $\beta l = 2\pi l / \lambda$ . Figure 9 shows a plot of  $X_{i,o}$  versus  $l$ .

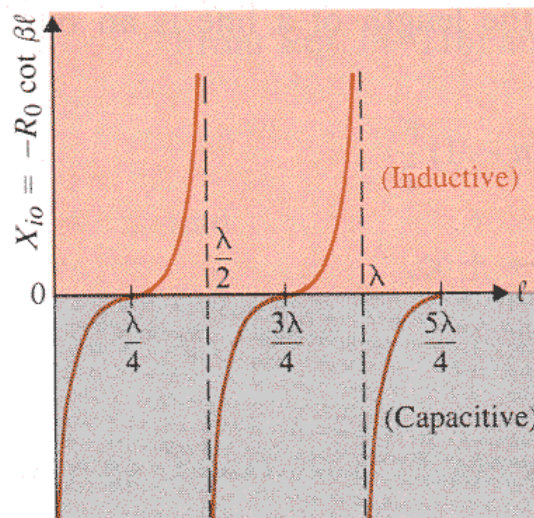


Figure 9: Input reactance of an open-circuited transmission line.

Note that at higher frequencies, it is impossible to obtain a true open circuit at the end of a transmission line, due to radiation from the open end.

(ii) *Lossless transmission line with a short circuit termination*

For an short circuit,  $Z_L = 0$ , and then

$$Z_{i,s} = jX_{i,s} = jZ_0 \tan \beta l = jZ_0 \tan(2\pi l / \lambda) \quad (46)$$

The input impedance of a short-circuited transmission line is also purely inductive or purely capacitive, depending on the length of the line. Figure 10 shows a plot of  $X_{i,s}$  versus  $l$ . Note that  $X_{i,s}$  is inductive when  $X_{i,o}$  is capacitive, and vice versa.

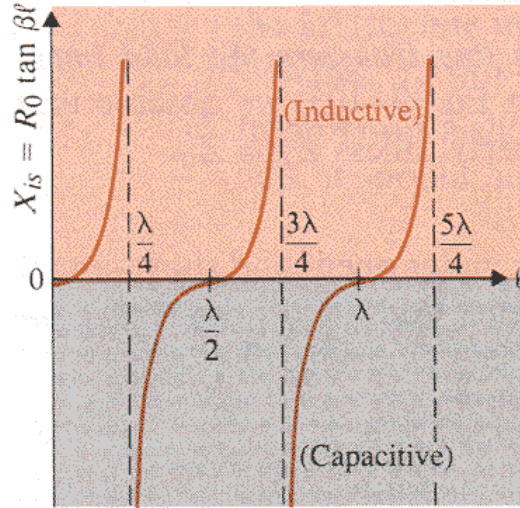


Figure 10: Input reactance of a short-circuited transmission line.

Also note that

$$Z_{i,o} Z_{i,s} = Z_0^2 \quad (47)$$

and

$$Z_{i,s} / Z_{i,o} = -\tan^2(\beta l). \quad (48)$$

We may use (47) and (48) to compute characteristic impedance  $Z_0$  and phase constant  $\beta$  of a transmission line of length  $l$ , provided that we have measured values of the short-circuit and open-circuit input impedance.

(iii) *Quarter-wave section with an arbitrary termination*

If a transmission line is terminated in an arbitrary load impedance  $Z_L$  and has a length of  $l = \lambda / 4$ , the input impedance is given by

$$\begin{aligned} Z_i &= Z_0 \frac{Z_L + jZ_0 \tan(\pi/2)}{Z_0 + jZ_L \tan(\pi/2)} \\ &= Z_0 \frac{Z_L / \tan(\pi/2) + jZ_0}{Z_0 / \tan(\pi/2) + jZ_L} \\ &= \frac{Z_0^2}{Z_L} \end{aligned} \quad (49)$$

The load impedance is therefore transformed into an impedance of  $Z_0^2 / Z_L$ . Note that the length of the transmission line is equal to a quarter of a transmission line wavelength, which is not necessarily the same as the free space wavelength.

(iv) *Half-wave section with an arbitrary termination*

If a transmission line is terminated in a load impedance  $Z_L$  and has a length of  $l = \lambda/2$ , the input impedance becomes

$$\begin{aligned} Z_i &= Z_0 \frac{Z_L + jZ_0 \tan(\pi)}{Z_0 + jZ_L \tan(\pi)} \\ &= Z_0 \frac{Z_L}{Z_0} \\ &= Z_L . \end{aligned} \quad (50)$$

The input impedance is therefore equal to the load impedance. From an impedance perspective, it is as if the transmission line was not present at all.

(v) *Transmission line terminated in a matched load*

If  $Z_L = Z_0$ , then for any length  $l$  we find that

$$\begin{aligned} Z_i &= Z_0 \frac{Z_0 + jZ_0 \tan(\beta l)}{Z_0 + jZ_0 \tan(\beta l)} \\ &= Z_0 . \end{aligned} \quad (51)$$

If a transmission line is matched, it therefore remains matched, irrespective of the length of transmission line inserted before the match.

**Example 2:** The open-circuit and short-circuit impedances measured at the input terminals of a lossless transmission line of length 1.5 m (which is less than a quarter wavelength) are  $-j54.6 \Omega$  and  $j103 \Omega$ , respectively. (a) Find  $Z_0$  and  $\beta$  of the line. (b) Without changing the operating frequency, find the input impedance of a short-circuited line that is twice the given length. (c) How long should the short-circuited line be in order for it to appear as an open circuit at the input terminals?

**Solution:** The given quantities are  $Z_{i,o} = -j54.6$ ,  $Z_{i,s} = j103$ ,  $l = 1.5$  m.

(a) Using (47) and (48), we find that

$$Z_0 = \sqrt{Z_{i,o} Z_{i,s}} = 75 \Omega$$

$$\beta = \frac{1}{l} \tan^{-1} \sqrt{-Z_{i,s} / Z_{i,o}} = 0.628 \text{ rad/m}$$

(b) For a line twice as long,  $l = 3$  m and  $\beta l = 1.884$  rad.

The input impedance is then given by (46):

$$Z_{i,s} = jZ_0 \tan \beta l = -j231 \Omega$$

(c) A short-circuited line which appears as an open circuit at the input terminals should be an odd multiple of a quarter wavelength long (see Figure 10), i.e.

$$l = \frac{2n+1}{4} \lambda, \quad n = 0, 1, 2, \dots$$

$$\text{where } \lambda = \frac{2\pi}{\beta} = 10 \text{ m}.$$