UNIT- III

TERMINOLOGY

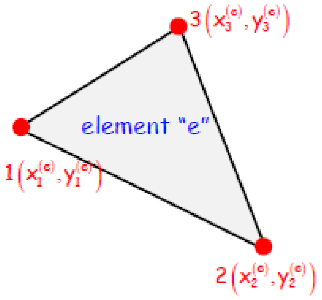
Natural Coordinates	Natural coordinate system is basically a local coordinate system which allows the specification of a point within the element by a set of dimensionless numbers whose magnitude never exceeds unity. This coordinate system is found to be very effective in formulating the element properties in finite element formulation. This system is defined in such that the magnitude at nodal points will have unity or zero or a convenient set of fractions. It also facilitates the integration to calculate element stiffness.
One Dimensional Line Elements	The line elements are used to represent spring, truss, beam like members for the finite element analysis purpose. Such elements are quite useful in analyzing truss, cable and frame structures. Such structures tend to be well defined in terms of the number and type of elements used. For example, to represent a truss member, a two node linear element is sufficient to get accurate results. However, three node line elements will be more suitable in case of analysis of cable structure to capture the nonlinear effects.
Two Dimensional Triangular Elements	The natural coordinate system for a triangular element is generally called as triangular coordinate system. The coordinate of any point P inside the triangle is x,y in Cartesian coordinate system. Here, three coordinates, L1, L2 and L3 can be used to define the location of the point in terms of natural coordinate system.
Shape Function using Area Coordinates	The interpolation functions for the triangular element are algebraically complex if expressed in Cartesian coordinates. Moreover, the integration required to obtain the element stiffness matrix is quite cumbersome. This will be discussed in details in next lecture. The interpolation function and subsequently the required integration can be obtained in a simplified manner by the concept of area coordinates.

Element Properties of Triangular Elements	The triangular element can be used to represent the arbitrary geometry much easily. On the other hand, rectangular elements, in general, are of limited use as they are not well suited for representing curved boundaries. However, an assemblage of rectangular and triangular element with triangular elements near the boundary can be very effective. Triangular elements may also be used in 3-dimensionalaxi-symmetric problems, plates and shell structures. The shape functions for triangular elements (linear, quadratic and cubic) with various nodes.
Displacement formulation	In displacement formulation, it is very important to approximate the variation of displacement in the element by suitable function. The interpolation function can be derived either using the Cartesian coordinate system or by the area coordinates.
Shape function using Cartesian coordinates	Polynomials are easiest way of mathematical operation for expressing variation of displacement. For example, the displacement variation within the element can be function of two dimensional plane stress/strain problems.
Higher Order Triangular Elements	Higher order elements are useful if the boundary of the geometry is curve in nature. For curved case, higher order triangular element can be suited effectively while generating the finite element mesh. Moreover, in case of flexural action in the member, higher order elements can produce more accurate results compare to those using linear elements. Various types of higher order triangular elements are used in practice.

Concepts

Three-Node Triangular Finite Element

The three-node triangular finite element is the first element to have been used in analysis. Since we have 3 nodes and want to introduce a polynomial interpolation, we can establish the shape functions in the same way as we did for one-dimensional elements:



$$T^{(e)}(x,y)=a_0^{(e)}+a_1^{(e)}x+a_2^{(e)}y \rightarrow T^{(e)}(x,y)=\begin{bmatrix}1 & x & y\end{bmatrix}\begin{Bmatrix}a_0^{(e)}\\ a_1^{(e)}\\ a_2^{(e)}\end{Bmatrix}=[p]\{a\}$$

$$\left.\begin{array}{l}\bullet\text{for } x=x_1^{(e)},y=y_1^{(e)} \rightarrow T^{(e)}\left(x=x_1^{(e)},y=y_1^{(e)}\right)=T_1^{(e)} \rightarrow a_0^{(e)}+a_1^{(e)}x_1^{(e)}+a_2^{(e)}y_1^{(e)}=T_1^{(e)}\ (1)\\ \bullet\text{for } x=x_2^{(e)},y=y_2^{(e)} \rightarrow T^{(e)}\left(x=x_2^{(e)},y=y_2^{(e)}\right)=T_2^{(e)} \rightarrow a_0^{(e)}+a_1^{(e)}x_2^{(e)}+a_2^{(e)}y_2^{(e)}=T_2^{(e)}\ (2)\\ \bullet\text{for } x=x_3^{(e)},y=y_3^{(e)} \rightarrow T^{(e)}\left(x=x_3^{(e)},y=y_3^{(e)}\right)=T_3^{(e)} \rightarrow a_0^{(e)}+a_1^{(e)}x_3^{(e)}+a_2^{(e)}y_3^{(e)}=T_3^{(e)}\ (3)\end{array}\right\}\rightarrow$$

$$\xrightarrow{\text{combine in a matrix equation}} \underbrace{\begin{bmatrix}1 & x_1^{(e)} & y_1^{(e)}\\ 1 & x_2^{(e)} & y_2^{(e)}\\ 1 & x_3^{(e)} & y_3^{(e)}\end{bmatrix}}_{[M^{(e)}]}\underbrace{\begin{Bmatrix}a_0^{(e)}\\ a_1^{(e)}\\ a_2^{(e)}\end{Bmatrix}}_{\{a^{(e)}\}}=\underbrace{\begin{Bmatrix}T_1^{(e)}\\ T_2^{(e)}\\ T_3^{(e)}\end{Bmatrix}}_{\{T^{(e)}\}}$$

It is worth mentioning here that, based on what we discussed above, the order of completeness in the polynomial approximation is equal to 1 (see the polynomial approximation and the terms in Pascal's triangle), which is adequate for the analysis of linear 2D heat conduction. We can now solve the resulting system of equations to obtain the vector $\{a^{(e)}\}$:

$$\left[M^{(e)}\right]\left\{a^{(e)}\right\}=\left\{T^{(e)}\right\} \rightarrow \left\{a^{(e)}\right\}=\left[M^{(e)}\right]^{-1}\left\{T^{(e)}\right\}$$

If we now plug the expression for $\{a^{(e)}\}$ in the equation $T^{(e)}(x,y)=[p]\{a\}$, we obtain:

$$T^{(e)}(x,y)=[p]\left[M^{(e)}\right]^{-1}\left\{T^{(e)}\right\}=\left[N^{(e)}\right]\left\{T^{(e)}\right\}=\begin{bmatrix}N_1^{(e)}(x,y) & N_2^{(e)}(x,y) & N_3^{(e)}(x,y)\end{bmatrix}\begin{Bmatrix}T_1^{(e)}\\ T_2^{(e)}\\ T_3^{(e)}\end{Bmatrix},$$

where $\left[N^{(e)}\right]=[p]\left[M^{(e)}\right]^{-1}$ is the shape function array for the three-node triangular element "e". If one does the math, the following expressions are obtained for the three shape functions:

$$N_1^{(e)}(x,y)=\frac{1}{2A^{(e)}}\left[x_2^{(e)}y_3^{(e)}-x_3^{(e)}y_2^{(e)}+\left(y_2^{(e)}-y_3^{(e)}\right)x+\left(x_3^{(e)}-x_2^{(e)}\right)y\right]$$
$$N_2^{(e)}(x,y)=\frac{1}{2A^{(e)}}\left[x_3^{(e)}y_1^{(e)}-x_1^{(e)}y_3^{(e)}+\left(y_3^{(e)}-y_1^{(e)}\right)x+\left(x_1^{(e)}-x_3^{(e)}\right)y\right]$$
$$N_3^{(e)}(x,y)=\frac{1}{2A^{(e)}}\left[x_1^{(e)}y_2^{(e)}-x_2^{(e)}y_1^{(e)}+\left(y_1^{(e)}-y_2^{(e)}\right)x+\left(x_2^{(e)}-x_1^{(e)}\right)y\right]$$

where

$$A^{(e)}=\frac{1}{2}\det\left(\left[M^{(e)}\right]\right)=\frac{1}{2}\left[\left(x_2^{(e)}y_3^{(e)}-x_3^{(e)}y_2^{(e)}\right)-\left(x_1^{(e)}y_3^{(e)}-x_3^{(e)}y_1^{(e)}\right)+\left(x_1^{(e)}y_2^{(e)}-x_2^{(e)}y_1^{(e)}\right)\right]$$

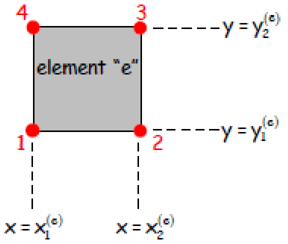
$$\frac{\partial N_1^{(e)}}{\partial x}=\frac{y_2^{(e)}-y_3^{(e)}}{2A^{(e)}},\quad \frac{\partial N_2^{(e)}}{\partial x}=\frac{y_3^{(e)}-y_1^{(e)}}{2A^{(e)}},\quad \frac{\partial N_3^{(e)}}{\partial x}=\frac{y_1^{(e)}-y_2^{(e)}}{2A^{(e)}}$$
$$\frac{\partial N_1^{(e)}}{\partial y}=\frac{x_3^{(e)}-x_2^{(e)}}{2A^{(e)}},\quad \frac{\partial N_2^{(e)}}{\partial y}=\frac{x_1^{(e)}-x_3^{(e)}}{2A^{(e)}},\quad \frac{\partial N_3^{(e)}}{\partial y}=\frac{x_2^{(e)}-x_1^{(e)}}{2A^{(e)}}$$

$$\left[B^{(e)}\right]=\frac{1}{2A^{(e)}}\begin{bmatrix} y_2^{(e)}-y_3^{(e)} & y_3^{(e)}-y_1^{(e)} & y_1^{(e)}-y_2^{(e)} \\ x_3^{(e)}-x_2^{(e)} & x_1^{(e)}-x_3^{(e)} & x_2^{(e)}-x_1^{(e)} \end{bmatrix}$$

It is worth mentioning that $[B^{(e)}]$ is CONSTANT!

Four-Node Rectangular Element

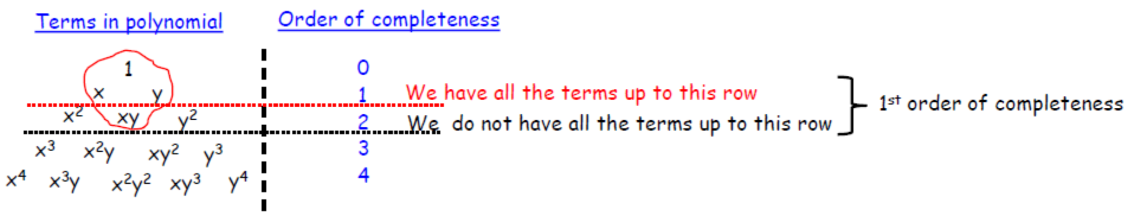
Let us now examine another relatively simple case of 2D finite element: The 4- Node rectangular finite element, as shown in the Figure below. Obviously, the polynomial element approximation will be having 4 terms:



$$T^{(e)}(x,y)=a_0^{(e)}+a_1^{(e)}x+a_2^{(e)}y+a_2^{(e)}x\cdot y\rightarrow T^{(e)}(x,y)=\begin{bmatrix}1 & x & y & x\cdot y\end{bmatrix}\begin{Bmatrix}a_0^{(e)}\\ a_1^{(e)}\\ a_2^{(e)}\\ a_3^{(e)}\end{Bmatrix}$$

$$\text{Thus, } T^{(e)}(x,y)=[p]\left\{a^{(e)}\right\},\ \text{ where } [p]=\begin{bmatrix}1 & x & y & x\cdot y\end{bmatrix},\ \left\{a^{(e)}\right\}=\begin{Bmatrix}a_0^{(e)}\\ a_1^{(e)}\\ a_2^{(e)}\\ a_3^{(e)}\end{Bmatrix}$$

Notice that for the fourth term in the polynomial we selected the monomial "xy", because it is "symmetric" in terms of x and y as shown in the figure below, the specific polynomial approximation includes all the terms ensuring first-order completeness, but only one of the terms corresponding to second-order completeness. Hence, the 4-Node Rectangular element approximation is complete to the first degree, which is adequate for the analysis of linear 2D heat conduction.



$$\left.\begin{array}{l} T^{(e)}\left(x=x_1^{(e)},y=y_1^{(e)}\right)=T_1^{(e)} \rightarrow a_0^{(e)}+a_1^{(e)}x_1^{(e)}+a_2^{(e)}y_1^{(e)}+a_3^{(e)}x_1^{(e)}\cdot y_1^{(e)}=T_1^{(e)} \\ T^{(e)}\left(x=x_2^{(e)},y=y_2^{(e)}\right)=T_2^{(e)} \rightarrow a_0^{(e)}+a_1^{(e)}x_2^{(e)}+a_2^{(e)}y_2^{(e)}+a_3^{(e)}x_2^{(e)}\cdot y_2^{(e)}=T_2^{(e)} \\ T^{(e)}\left(x=x_3^{(e)},y=y_3^{(e)}\right)=T_3^{(e)} \rightarrow a_0^{(e)}+a_1^{(e)}x_3^{(e)}+a_2^{(e)}y_3^{(e)}+a_3^{(e)}x_3^{(e)}\cdot y_3^{(e)}=T_3^{(e)} \\ T^{(e)}\left(x=x_4^{(e)},y=y_4^{(e)}\right)=T_4^{(e)} \rightarrow a_0^{(e)}+a_1^{(e)}x_4^{(e)}+a_2^{(e)}y_4^{(e)}+a_3^{(e)}x_4^{(e)}\cdot y_4^{(e)}=T_4^{(e)} \end{array}\right\} \rightarrow \begin{bmatrix} 1 & x_1^{(e)} & y_1^{(e)} & x_1^{(e)}\cdot y_1^{(e)} \\ 1 & x_2^{(e)} & y_2^{(e)} & x_2^{(e)}\cdot y_2^{(e)} \\ 1 & x_3^{(e)} & y_3^{(e)} & x_3^{(e)}\cdot y_3^{(e)} \\ 1 & x_4^{(e)} & y_4^{(e)} & x_4^{(e)}\cdot y_4^{(e)} \end{bmatrix} \begin{Bmatrix} a_0^{(e)} \\ a_1^{(e)} \\ a_2^{(e)} \\ a_3^{(e)} \end{Bmatrix} = \begin{Bmatrix} T_1^{(e)} \\ T_2^{(e)} \\ T_3^{(e)} \\ T_4^{(e)} \end{Bmatrix}$$

$$[M]=\begin{bmatrix} 1 & x_1^{(e)} & y_1^{(e)} & x_1^{(e)}\cdot y_1^{(e)} \\ 1 & x_2^{(e)} & y_2^{(e)} & x_2^{(e)}\cdot y_2^{(e)} \\ 1 & x_3^{(e)} & y_3^{(e)} & x_3^{(e)}\cdot y_3^{(e)} \\ 1 & x_4^{(e)} & y_4^{(e)} & x_4^{(e)}\cdot y_4^{(e)} \end{bmatrix}$$

We obtain: $[M]\left\{a^{(e)}\right\}=\left\{T^{(e)}\right\} \rightarrow \left\{a^{(e)}\right\}=[M]^{-1}\left\{T^{(e)}\right\}$

$$T^{(e)}(x,y)=[p]\left\{a^{(e)}\right\}=[p][M]^{-1}\left\{T^{(e)}\right\}=\left[N^{(e)}\right]\left\{T^{(e)}\right\},$$

where $\left[N^{(e)}\right]=[p][M]^{-1}$ is the shape function row array for element "e".

The shape functions are given by:

$$N_1^{(e)}(x,y)=\frac{x-x_2^{(e)}}{x_1^{(e)}-x_2^{(e)}}\cdot\frac{y-y_2^{(e)}}{y_1^{(e)}-y_2^{(e)}},\ N_2^{(e)}(x,y)=\frac{x-x_1^{(e)}}{x_2^{(e)}-x_1^{(e)}}\cdot\frac{y-y_2^{(e)}}{y_1^{(e)}-y_2^{(e)}},\ N_3^{(e)}(x,y)=\frac{x-x_1^{(e)}}{x_2^{(e)}-x_1^{(e)}}\cdot\frac{y-y_1^{(e)}}{y_2^{(e)}-y_1^{(e)}},\ N_4^{(e)}(x,y)=\frac{x-x_2^{(e)}}{x_1^{(e)}-x_2^{(e)}}\cdot\frac{y-y_1^{(e)}}{y_2^{(e)}-y_1^{(e)}}$$

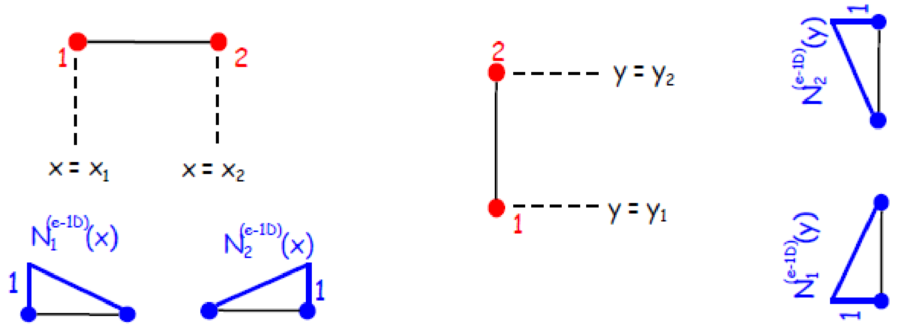
If we now notice that the area of the element, $A^{(e)}$, is given by:

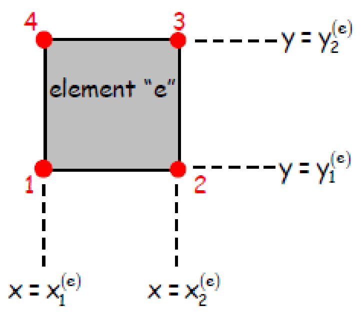
$$A^{(e)}=\left(x_2^{(e)}-x_1^{(e)}\right)\left(y_2^{(e)}-y_1^{(e)}\right)=\left(x_1^{(e)}-x_2^{(e)}\right)\left(y_1^{(e)}-y_2^{(e)}\right)$$

and that the denominator in each shape function is equal to $A^{(e)}$ or $-A^{(e)}$, we obtain:

$$N_1^{(e)}(x,y)=\frac{\left(x-x_2^{(e)}\right)\left(y-y_2^{(e)}\right)}{A^{(e)}},\ N_2^{(e)}(x,y)=-\frac{\left(x-x_1^{(e)}\right)\left(y-y_2^{(e)}\right)}{A^{(e)}},\ N_3^{(e)}(x,y)=\frac{\left(x-x_1^{(e)}\right)\left(y-y_1^{(e)}\right)}{A^{(e)}},\ N_4^{(e)}(x,y)=-\frac{\left(x-x_2^{(e)}\right)\left(y-y_1^{(e)}\right)}{A^{(e)}}$$

Alternatively, the shape functions for a rectangular element aligned with axes x and y can be obtained as **"tensor products" of the one-dimensional 2-node element shape functions**:





$$N_1^{(e)}(x,y)=\frac{x-x_2^{(e)}}{x_1^{(e)}-x_2^{(e)}}\cdot\frac{y-y_2^{(e)}}{y_1^{(e)}-y_2^{(e)}}=N_1^{(e-1D)}(x)\cdot N_1^{(e-1D)}(y)$$

$$N_2^{(e)}(x,y)=\frac{x-x_1^{(e)}}{x_2^{(e)}-x_1^{(e)}}\cdot\frac{y-y_2^{(e)}}{y_1^{(e)}-y_2^{(e)}}=N_2^{(e-1D)}(x)\cdot N_1^{(e-1D)}(y)$$

$$N_3^{(e)}(x,y)=\frac{x-x_1^{(e)}}{x_2^{(e)}-x_1^{(e)}}\cdot\frac{y-y_1^{(e)}}{y_2^{(e)}-y_1^{(e)}}=N_2^{(e-1D)}(x)\cdot N_2^{(e-1D)}(y)$$

$$N_4^{(e)}(x,y)=\frac{x-x_2^{(e)}}{x_1^{(e)}-x_2^{(e)}}\cdot\frac{y-y_1^{(e)}}{y_2^{(e)}-y_1^{(e)}}=N_1^{(e-1D)}(x)\cdot N_2^{(e-1D)}(y)$$

Note that the "exterior product" approach can only be used for rectangular elements, the sides of which are aligned with the two Cartesian coordinate axes, x and y. We can now calculate the derivatives of the shape functions with x and y to obtain the $[B^{(e)}]$ array:

$$\frac{\partial N_1^{(e)}}{\partial x}=\frac{y-y_2^{(e)}}{A^{(e)}},\ \frac{\partial N_2^{(e)}}{\partial x}=\frac{y_2^{(e)}-y}{A^{(e)}},\ \frac{\partial N_3^{(e)}}{\partial x}=\frac{y-y_1^{(e)}}{A^{(e)}},\ \frac{\partial N_4^{(e)}}{\partial x}=\frac{y_1^{(e)}-y}{A^{(e)}}$$

$$\frac{\partial N_1^{(e)}}{\partial y}=\frac{x-x_2^{(e)}}{A^{(e)}},\ \frac{\partial N_2^{(e)}}{\partial y}=\frac{x_1^{(e)}-x}{A^{(e)}},\ \frac{\partial N_3^{(e)}}{\partial y}=\frac{x-x_1^{(e)}}{A^{(e)}},\ \frac{\partial N_4^{(e)}}{\partial y}=\frac{x_2^{(e)}-x}{A^{(e)}}$$

$$\frac{\partial N_1^{(e)}}{\partial x}=\frac{y-y_2}{A^{(e)}},\ \frac{\partial N_2^{(e)}}{\partial x}=\frac{y_2-y}{A^{(e)}},\ \frac{\partial N_3^{(e)}}{\partial x}=\frac{y-y_1}{A^{(e)}},\ \frac{\partial N_4^{(e)}}{\partial x}=\frac{y_1-y}{A^{(e)}}$$

$$\frac{\partial N_1^{(e)}}{\partial y}=\frac{x-x_2}{A^{(e)}},\ \frac{\partial N_2^{(e)}}{\partial y}=\frac{x_1-x}{A^{(e)}},\ \frac{\partial N_3^{(e)}}{\partial y}=\frac{x-x_1}{A^{(e)}},\ \frac{\partial N_4^{(e)}}{\partial y}=\frac{x_2-x}{A^{(e)}}$$

$$\left[B^{(e)}\right]=\frac{1}{A^{(e)}}\begin{bmatrix} y-y_2^{(e)} & y_2^{(e)}-y & y-y_1^{(e)} & y_1^{(e)}-y \\ x-x_2^{(e)} & x_1^{(e)}-x & x-x_1^{(e)} & x_2^{(e)}-x \end{bmatrix}$$

Calculation of Element Conductance Matrices and Nodal Flux Vectors

We have shown how to establish the various arrays for the two different types of 2-dimensional finite elements. All that remains to do is show how to calculate the conductance arrays and nodal flux vectors:

$$\left[k^{(e)}\right]=\iint_{\Omega^{(e)}}\left[B^{(e)}\right]^T\left[D^{(e)}\right]\left[B^{(e)}\right]dV \text{ and } \left\{f^{(e)}\right\}=\iint_{\Omega^{(e)}}\left[N^{(e)}\right]^T s\cdot dV-\int_{\Gamma_q}\left[N^{(e)}\right]^T\bar{q}ds$$

The right-hand side vector can be separated into two terms, one corresponding to the contribution from the source term and the other to the contribution of the boundary term:

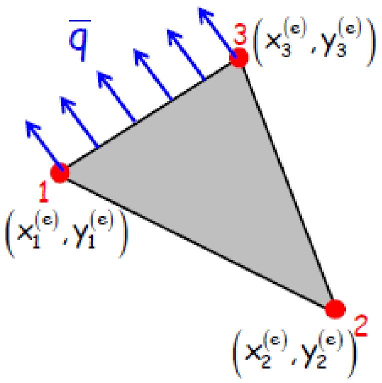
$$\left\{f^{(e)}\right\}=\left\{f_\Omega^{(e)}\right\}+\left\{f_{\Gamma_q}^{(e)}\right\},\text{ where }\left\{f_\Omega^{(e)}\right\}=\iint_{\Omega^{(e)}}\left[N^{(e)}\right]^T s\cdot dV \text{ and } \left\{f_{\Gamma_q}^{(e)}\right\}=-\int_{\Gamma_q}\left[N^{(e)}\right]^T\bar{q}ds$$

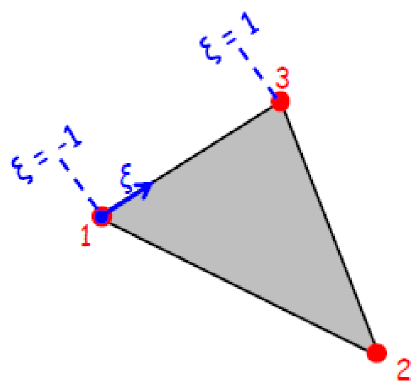
Calculation of [k(e)], {f(e)} for three-node triangular elements

For three-node triangular finite elements we saw that the $[B^{(e)}]$ array is constant inside the element. If the material conductivity array, $\left[D^{(e)}\right]=\begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{xx} \end{bmatrix}^{(e)}$, is constant, then the element's conductance matrix can be easily obtained, since $[B^{(e)}]$, $[D^{(e)}]$ and $[B^{(e)}]^T$ can be taken outside of the integral!

$$\left[k^{(e)}\right]=\iint_{\Omega^{(e)}}\left[B^{(e)}\right]^T\left[D^{(e)}\right]\left[B^{(e)}\right]dV=\left[B^{(e)}\right]^T\left[D^{(e)}\right]\left[B^{(e)}\right]\iint_{\Omega^{(e)}}dV=\left[B^{(e)}\right]^T\left[D^{(e)}\right]\left[B^{(e)}\right]\cdot A^{(e)}$$

Of course, the terms for {f(e)} will still include matrices which are functions of x and y in the integral… If a side of a triangular element constitutes a segment of the natural boundary, we need to also calculate the contribution of the boundary term to the nodal flux vector of the element. To do so, we parameterize the boundary segment. Since for a 2-dimensional triangular element with three nodes the sides of the element are lines, we can parameterize the boundary segment using a single parameter, ξ, with -1 ≤ ξ ≤ 1, and express the values of x and y along the boundary segment as functions of ξ, as shown for example for the following case:





In this case, segment (1-3) is a boundary segment, so we can introduce the parameterization shown above. Now, we can express the values of x and y along the natural boundary segment as functions of ξ:

$$\left.\begin{aligned} x(\xi) &= \frac{x_1^{(e)} + x_3^{(e)}}{2} + \frac{x_3^{(e)} - x_1^{(e)}}{2}\xi \quad (1) \\ y(\xi) &= \frac{y_1^{(e)} + y_3^{(e)}}{2} + \frac{y_3^{(e)} - y_1^{(e)}}{2}\xi \quad (2) \end{aligned}\right\}, \ -1 \le \xi \le 1$$

We can also express the values of the 3 shape functions along the natural boundary with respect to the single parameter ξ:

$$N_1^{(e)}(x(\xi), y(\xi)) = \frac{1}{2A^{(e)}}\left[x_2^{(e)}y_3^{(e)} - x_3^{(e)}y_2^{(e)} + \left(y_2^{(e)} - y_3^{(e)}\right)x(\xi) + \left(x_3^{(e)} - x_2^{(e)}\right)y(\xi)\right]$$

$$N_2^{(e)}(x(\xi), y(\xi)) = \frac{1}{2A^{(e)}}\left[x_3^{(e)}y_1^{(e)} - x_1^{(e)}y_3^{(e)} + \left(y_3^{(e)} - y_1^{(e)}\right)x(\xi) + \left(x_1^{(e)} - x_3^{(e)}\right)y(\xi)\right]$$

$$N_3^{(e)}(x(\xi), y(\xi)) = \frac{1}{2A^{(e)}}\left[x_1^{(e)}y_2^{(e)} - x_2^{(e)}y_1^{(e)} + \left(y_1^{(e)} - y_2^{(e)}\right)x(\xi) + \left(x_2^{(e)} - x_1^{(e)}\right)y(\xi)\right]$$

From a fundamental theorem of calculus, the differential length "dS" in the boundary integral is equal to:

$$dS = \sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2}\, d\xi = \frac{L_{31}}{2} d\xi$$

where $L_{31} = \sqrt{\left(x_3^{(e)} - x_1^{(e)}\right)^2 + \left(y_3^{(e)} - y_1^{(e)}\right)^2}$ is the length of the boundary segment!

We can now go ahead and evaluate the integral…

$$\left\{f_{\Gamma_q}^{(e)}\right\} = -\int_{\Gamma_q}\left[N^{(e)}\right]^T \bar{q}\, dS = -\int_{-1}^{1}\begin{Bmatrix} N_1^{(e)}(x(\xi), y(\xi)) \\ N_2^{(e)}(x(\xi), y(\xi)) \\ N_3^{(e)}(x(\xi), y(\xi)) \end{Bmatrix}\bar{q}\frac{L_{31}}{2}d\xi = -\begin{Bmatrix} \int_{-1}^{1} N_1^{(e)}(x(\xi), y(\xi))\bar{q}\frac{L_{31}}{2}d\xi \\ \int_{-1}^{1} N_2^{(e)}(x(\xi), y(\xi))\bar{q}\frac{L_{31}}{2}d\xi \\ \int_{-1}^{1} N_3^{(e)}(x(\xi), y(\xi))\bar{q}\frac{L_{31}}{2}d\xi \end{Bmatrix}$$

Important Questions

1. Explain about Generation of element stiffness matrices for 3-node triangular elements.

2. Explain about Generation of nodal load matrices for 3-node triangular elements.

3. Explain about Generation of element stiffness matrices for 4- node rectangular elements.

4. Explain about Generation of nodal load matrices for 4-node rectangular elements.

5. Write about element stiffness & nodal load matrices for 3-node triangular elements.

6. Write about element stiffness & nodal load matrices for 4- node rectangular elements.