MATHEMATICS-III

Unit-I:

Basic Terms and Definitions:

Matrix	A system of m ×n numbers arranged in the form of an ordered set of m horizontal lines called rows & n vertical lines called columns is called an m× n matrix. The matrix of order m ×n is written as $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{1j} & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{2j} & a_{2n} \\ \text{-------} & \text{-----} & \text{-------} & \text{-----} & \text{-----} \\ a_{i1} & a_{i2} & a_{i3} & a_{ij} & a_{in} \\ a_{m1} & a_{m2} & a_{m3} & a_{mj} & a_{mn} \end{bmatrix}_{m \times n}$
Rectangular matrix	Any mxn Matrix where m≠n is called rectangular matrix Eg: $A_{2\times1}=\begin{bmatrix}1\\5\end{bmatrix}$
Column Matrix	It is a matrix in which there is only one column Eg: $A_{3\times1}=\begin{bmatrix}1\\3\\8\end{bmatrix}$
Row Matrix	It is a matrix in which there is only one row Eg: $A_{1\times3}=\begin{bmatrix}4 & -1 & 3\end{bmatrix}$
Square Matrix	It is a matrix in which number of rows equals the number of columns i.e its order is n x n. Eg: $A_{2\times2}=\begin{bmatrix}1 & 4\\-2 & 7\end{bmatrix}$
Diagonal Matrix	It is a square matrix in which all non-diagonal elements are zero Eg: $A_{3\times3}=\begin{bmatrix}1 & 0 & 0\\0 & 6 & 0\\0 & 0 & 2\end{bmatrix}$
Scalar Matrix	It is a square diagonal matrix in which all diagonal elements are equal Eg: $A_{3\times3}=\begin{bmatrix}4 & 0 & 0\\0 & 4 & 0\\0 & 0 & 4\end{bmatrix}$
Unit Matrix:	It is a scalar matrix with diagonal elements as unity Eg: $A_{3\times3}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$
Upper Triangular Matrix	It is a square matrix in which all the elements below the principle diagonal are zero Eg: $A_{3\times3}=\begin{bmatrix}1 & 3 & 0\\0 & 7 & -2\\0 & 0 & 4\end{bmatrix}$
Lower Triangular Matrix	It is a square matrix in which all the elements above the principle diagonal are zero Eg: $A_{3\times3}=\begin{bmatrix}1 & 0 & 0\\-3 & -5 & 0\\4 & 1 & 2\end{bmatrix}$
Transpose of Matrix	It is a matrix obtained by interchanging rows into columns or columns into rows

	Eg: $A_{2\times3}=\begin{bmatrix}1 & 3 & -2\\ 0 & 5 & 7\end{bmatrix}$ $A^T = Transpose\ of A = \begin{bmatrix}1 & 0\\ 3 & 5\\ -2 & 7\end{bmatrix}$ A^T and B^T be the transposes of A and B repectively, then (i) $(A^1)^1 = A$ (ii) $(A+B)^1 = A^1+B^1$ (iii) $(KA)^1 = KA^1$, K is a scalar (iv) $(AB)^1= B^1A^1$ Since $A^T = A^I$
Symmetric Matrix	If for a square matrix A, A =A^T then A is symmetric Eg: $A_{3\times3}=\begin{bmatrix}1 & 3 & 5\\ 3 & 4 & 1\\ 5 & 1 & 2\end{bmatrix}$
Skew Symmeric Matrix	If for a square matrix A, -A= A^T then it is skew -symmetric matrix. Eg: $A_{3\times3}=\begin{bmatrix}0 & 3 & 5\\ -3 & 0 & 1\\ -5 & -1 & 0\end{bmatrix}$ Note : For a skew Symmetric matrix, diagonal elements are zero.
Determinant of a Matrix	Let A be a square matrix then | A| =determinant of A i.e det A=| A| If (i) then $\|A\|\neq 0$ matrix , A is called as non-singular and If (i) then |A|=0 matrix, A is singular. Note : for non-singular matrix A^{-1}exists
Inverse of a Matrix	: Let A be any square matrix, then a matrix B, if exists such that AB = BA =I then B is called inverse of A and is denoted by A^{-1}.
Adjoint of a matrix	Let A be a square matrix of order n. The transpose of the matrix got from ABy replacing the elements of A by the corresponding co-factors is called the adjoint of A and is denoted by adj A.
The conjugate of a matrix	: The matrix obtained from any given matrix A, on replacing its elements by corresponding conjugate complex numbers is called the conjugate of A and is denoted *by* $\bar{A}$

	Note : if $\bar{A}$ and $\bar{B}$ be the conjugates of A and B respectively then, (i) $\overline{(\bar{A})} = A$ (ii) $\overline{(A+B)} = \bar{A}+\bar{B}$ (iii) $\overline{(KA)} = \bar{K}\bar{A}$, K is a any complex number (iv) $\overline{(AB)} = \bar{B}\,\bar{A}$ Eg ; if A= $\begin{bmatrix} 2 & 3i & 2-5i \\ -i & 0 & 4i+3 \end{bmatrix}_{2x3}$ $then\ \bar{A} = \begin{bmatrix} 2 & -3i & 2+5i \\ i & 0 & -4i+3 \end{bmatrix}_{2x3}$
. Idempotent matrix	If A is a square matrix such that A^2= A then 'A' is called idempotent matrix
Nilpotent Matrix	If A is a square matrix such that A^m=0 where m is a +ve integer then A is called nilpotent matrix Note : If m is least positive integer such that A^m = 0 then A is called nilpotent of index m
Involutary	If A is a square matrix such that A^2 = I then A is called involuntary matrix
. Orthogonal Matrix	A square matrix A is said to be orthogonal if AA^T = A^TA = I
Sub – Matrix	Any matrix obtained by deleting some rows or columns or both of a given matrix is called is submatrix. E.g: Let A=$\begin{bmatrix} 1 & 4 & 9 & 2 \\ -2 & 7 & 0 & 5 \\ 6 & 5 & 3 & 4 \end{bmatrix}$, Then $\begin{bmatrix} -2 & 7 & 0 \\ 6 & 5 & 3 \end{bmatrix}$ is a sub matrix of A obtained by deleting first row and fourth column of A
Minor of a matrix	Let A be an mxn matrix. The determinant of a square sub matrix of A is called a minor of the matrix. Note: If the order of the square sub matrix is 't' then its determinant is called a minor of order 't' Eg: A = $\begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}_{4X3}$ be a matrix $\rightarrow B = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ is a sub-matrix of order '2' $\lvert B \rvert$ = 2-3 = -1 is a minor of order '2'
Rank of a matrix	Let A be mxn matrix. If A is a null matrix, we define its rank to be '0'. If A is a non-zero matrix, we say that r is the rank of A if (i) Every (r+1)th order minor of A is '0' (zero) & (ii) At least one rth order minor of A which is not zero. Note: 1. It is denoted by ρ(A) 2. Rank of a matrix is unique. 3. Every matrix will have a rank. 4. If A is a matrix of order mxn, Rank of A ≤ min(m,n)

	5. If ρ(A) = r then every minor of A of order r+1, or more is zero. 6. Rank of the Identity matrix In is n. 7. If A is a matrix of order n and A is non-singular then ρ(A) = n Important Note: 1. The rank of a matrix is ≤r if all minors of (r+1)th order are zero 2. The rank of a matrix is ≥r, if there is at least one minor of order ‘r’ which is not equal to zero.
Characteristic Equation:	The equation $\lvert A-\lambda I\rvert = 0$ is called the characterstic equation of the matrix
Characteristic Polynomial:	The determinant $\lvert A-\lambda I\rvert$ when expanded will give a polynomial, which we call as characterstic polynomial of matrix A
Cayley-Hamilton Theorem:	Every square matrix satisfies its own characteristic equation
Eigen Values And Eigen Vectors	1. Solving $\lvert A-\lambda I\rvert = 0$, we get n roots for λ and these roots are called characteristic roots or eigen values or latent values of the matrix 2. Corresponding to each value of λ the equation AX =Λx has a non-zero solution vector X 3. If X_r be the non-zero vector satisfying AX=Λx ,when λ=λ_r, X_r is said to be the latent vector or eigen vector of a matrix A corresponding to λ_r
Properties Of Eigen Values And Eigen Vectors	1.The sum of the eigen values of a square matrix is equal to its trace and product of the eigen values is equal to its determinant. 2. If is an eigen value of A corresponding to the eigen vector X, then is eigen value An corresponding to the eigen vector X. 3. A Square matrix A and its transpose AT have the same eigen values 4. If A and B are n-rowed square matrices and If A is invertible show that A-1B and B A-1 have same eigen values 5. If$\lambda_1, \lambda_2, \lambda_3 \dots\dots\dots, \lambda_n$ are the eigen values of a matrix A then k λ_1, k λ_2, ….. k λ_n are the eigen value of the matrix KA, where K is a non-zero scalar. 6. If λ is an eigen values of the matrix A then λ+K is an eigen value of the matrix A+KI 7. If λ is an eigen value of a non-singular matrix A corresponding to the eigen vector X, then λ^{-1} is an eigen value of A–1 and corresponding eigen vector X itself. 8.If λ is an eigen value of a non singular matrix A ,then $\frac{\lvert A\rvert}{\lambda}$ is an eigen value of the matrix $AdjA$ 9. Suppose that A and P be square matrices of order n such that P is non singular. Then A and P-1AP have the same eigen values. 10. The eigen values of a triangular matrix are just the diagonal elements of the matrix 11. The eigen values of a real symmetric matrix are always real
Conjugate Of A Matrix	If the elements of the matrix A are replaced by their conjugates then the resulting matrix is defined as the conjugate of the given matrix ,we denote it with $\bar{A}$

	e.g If A= $\begin{bmatrix} 2+3i & 5 \\ 6-7i & -5+i \end{bmatrix}$ then $\overline{A}=\begin{bmatrix} 2-3i & 5 \\ 6+7i & -5-i \end{bmatrix}$
The Transpose Of The Conjugate Of A Square Matrix	If A is a square matrix and its conjugate is $\overline{A}$, then the transpose of $\overline{A}$ is $\left(\overline{A}\right)^T$. It can be easily seen that $\left(\overline{A}\right)^T=\overline{A^T}$ It is denoted by A^θ $A^\theta=\left(\overline{A}\right)^T=\overline{A^T}$ Note: If A^θ and B^θ be the transposed conjugates of A and B respectively, then i) $\left(A^\theta\right)^\theta=A$ ii) $\left(A\pm B\right)^\theta=A^\theta\pm B^\theta$ iii) $\left(KA\right)^\theta=\overline{K}A^\theta$ iv) $\left(AB\right)^\theta=B^\theta A^\theta$
Hermitian Matrix	A square matrix A such that $\overline{A}=A^T$ (or) $\left(\overline{A}\right)^T$ =A is called a hermitian matrix e.g A= $\begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$ then $\overline{A}=\begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ and $A^\theta=\begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$ Here $\left(\overline{A}\right)^T$ =A, Hence A is called Hermitian Note: 1) The element of the principal diagonal of a Hermitian matrix must be real 2) A hermitian matrix over the field of real numbers is nothing but a real symmetric.
Skew- Hermitian Matrix	A square matrix A such that $A^T=-\overline{A}$ (or) $\left(\overline{A}\right)^T$ =-A is called a Skew-Hermitian matrix e.g. Let A= $\begin{bmatrix} -3i & 2+i \\ -2+i & -i \end{bmatrix}$ then $\overline{A}=\begin{bmatrix} 3i & 2-i \\ -2-i & i \end{bmatrix}$ and $\left(\overline{A}\right)^T=\begin{bmatrix} 3i & -2-i \\ 2-i & i \end{bmatrix}$ $\therefore \left(\overline{A}\right)^T$ =-A Note: 1) The elements of the leading diagonal must be zero (or) all are purely imaginary 2) A skew-Hermitian matrix over the field of real numbers is nothing but a real skew-symmetric matrix.
Unitary matrix	A square matrix A such that $\left(\overline{A}\right)^T=A^{-1}$ i.e $\left(\overline{A}\right)^T$ A=A$\left(\overline{A}\right)^T$ =I If A^θ A=I then A is called Unitary matrix
Modal and spectral matrices	The matrix P in the above result which diagonalize the square matrix A is called modal matrix of A and the resulting diagonal matrix D is known as spectral matrix
Quadratic form:	A homogeneous polynomial of second degree in any number of variables is called a quadratic form

	Eg: $2x_1^2+3x_2^2-x_3^2+4x_1x_2+5x_1x_3-6x_2x_3$ is a quadratic form in three variables
Rank of the quadratic form	The number of square terms in the canonical form is the rank (r) of the quadratic form
Index of the quadratic form	The number of positive square terms in the canonical form is called the index (s) of the quadratic form
Signature of the quadratic form	The difference between the number of positive and negative square terms = s – (r-s) = 2s-r, is called the signature of the quadratic form
Positive definite	if all the eigen values are positive numbers
Negative definite	if all the eigen values are negative numbers
Positive Semi-definite	if all the eigen values are greater than or equal to zero and at least one eigen value is zero
Negative Semi-definite	if all the eigen values are less than or equal to zero and at least one eigen value is zero
Indefinite	if A has both positive and negative eigen values

Concepts

1. Find the rank of the given matrix $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

Soln: Given matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

$\rightarrow$ det A = 1(48-40)-2(36-28)+3(30-28)

= 8-16+6 = -2 $\neq$ 0

We have minor of order 3

$\rho(A) = 3$

Hence rank of the given matrix is '3'.

*** Elementary Transformations on a Matrix:**

i). Interchange of i^{th} row and j^{th} row is denoted by $R_i \leftrightarrow R_j$

(ii). If i^{th} row is multiplied with k then it is denoted by $R_i \to K R_i$

(iii). If all the elements of i^{th} row are multiplied with k and added to the corresponding elements of j^{th} row then it is denoted by $R_j \to R_j + KR_i$

<u>**Note:**</u> 1. The corresponding column transformations will be denoted by writing 'c'. i.e

$c_i \leftrightarrow c_j$, $c_i \to k\, c_j$ $c_j \to c_j + kc_i$

2. The elementary operations on a matrix do not change its rank.

Equivalence of Matrices: If B is obtained from A after a finite number of elementary transformations on A, then B is said to be equivalent to A.

It is denoted as B~A.

Note : 1. If A and B are two equivalent matrices, then rank A = rank B.

2. If A and B have the same size and the same rank, then the two matrices are equivalent.

Echelon form of a matrix:

A matrix is said to be in Echelon form, if

(i). Zero rows, if any exists, they should be below the non-zero row.

(ii). The first non-zero entry in each non-zero row is equal to '1'.

(iii). The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

<u>**Note:**</u> 1. The number of non-zero rows in echelon form of A is the rank of 'A'.

2. The rank of the transpose of a matrix is the same as that of original matrix.
3. The condition (ii) is optional.

Eg: 1. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is a row echelon form.

2. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is a row echelon form.

Problems:

1. **Find the rank of the matrix $A = \begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$ by reducing it to Echelon form.**

sol: Given $A = \begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$

Applying row transformations on A.

$A \sim \begin{bmatrix} 1 & -3 & -1 \\ 3 & -2 & 4 \\ 2 & 3 & 7 \end{bmatrix}$ $R_1 \leftrightarrow R_3$

$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 7 & 7 \\ 0 & 9 & 9 \end{bmatrix}$ $R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 2R_1$

$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $R_2 \to R_2/7, R_3 \to R_3/9$

$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $R_3 \to R_3 - R_2$

This is the Echelon form of matrix A.

The rank of a matrix A.

= Number of non – zero rows =2

2. **For what values of k the matrix $\begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & -2 \\ 9 & 9 & k & 3 \end{bmatrix}$ has rank '3'.**

Sol: The given matrix is of the order 4x4

If its rank is 3 ⇨ det A =0

$A = \begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & -2 \\ 9 & 9 & k & 3 \end{bmatrix}$

Applying $R_2 \to 4R_2 - R_1$, $R_3 \to 4R_3 - kR_1$, $R_4 \to 4R_4 - 9R_1$

We get $A \sim \begin{bmatrix} 4 & 4 & -3 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 8-4k & 8+3k & 8-k \\ 0 & 0 & 4k+27 & 3 \end{bmatrix}$

Since Rank A = 3 ⇨ det A =0

$$\Rightarrow 4\begin{vmatrix} 0 & -1 & -1 \\ 8-4k & 8+3k & 8-k \\ 0 & 4k+27 & 3 \end{vmatrix} = 0$$

⇨ 1[(8-4k)3]-1(8-4k)(4k+27)] = 0

⇨ (8-4k) (3-4k-27) = 0

⇨ (8-4k)(-24-4k) =0

⇨ (2-k)(6+k)=0

K=2 or k=-6

Normal Form:

Every mxn matrix of rank r can be reduced to the form $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ (or) (I_r) (or) $\begin{pmatrix} I_r & \\ 0 & \end{pmatrix}$ (or) $\begin{pmatrix} I_r & 0 \\ & \end{pmatrix}$ by a finite number of elementary transformations, where I_r is the r – rowed unit matrix.

Note: 1. If A is an mxn matrix of rank r, there exists non-singular matrices P and Q such that PAQ = $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$

2.Normal form another name is "canonical form"

e.g: By reducing the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$ into normal form, find its rank.

Sol: Given A = $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & 5 \\ 0 & -6 & -4 & -22 \end{bmatrix} R_2 \rightarrow R_2 - 2R_1$$

$R_3 \rightarrow R_3 - 3R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & 3 & 2 & 11 \end{bmatrix} R_3 \rightarrow R_3/-2$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix} R_3 \rightarrow R_3+R_2$$

$$A \sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix} c_2 \rightarrow c_2 - 2c_1,\ c_3 \rightarrow c_3 - 3c_1,\ c_4 \rightarrow c_4 - 4c_1$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 18 \end{bmatrix} c_3 \to 3c_3 - 2c_2,\ c_4 \to 3c_4 - 5c_2$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} c_2 \to c_2/-3,\ c_4 \to c_4/18$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} c_4 \leftrightarrow c_3$$

This is in normal form $[I_3\ 0]$

Hence Rank of A is '3'.

To find two non singular matrices P and Q such that PAQ is in the normal form

Every matrix $A_{m\times n}$ having the rank r can be reduced into the following form

PAQ=(I_r) or $\begin{pmatrix} I_r \\ 0 \end{pmatrix}$ or $(I_r \quad 0)$ or $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$where P,Q are non singular matrices of orders m and n by using both row and column operations .

Procedure:

1.Write $A_{m\times n} = I_{m\times m}$ A $I_{n\times n}$

2.Reduce the matrix A on LHS to its normal form by using both row and column operations.

3>Every elementary row transformation on A must be accompanied by the same transformation on the pre factor on RHS.

4>Every elementary column transformation on Amust be accompanied by the same transformation on the post factor on RHS.

Problems

Find the non-singular matrices P and Q such that PAQ is in normal and hence find the rank of A.

i) $A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 4 & -1 \\ 1 & 5 & -4 \end{bmatrix}$

Solution: Consider

$A = I_3 A I_3$

$$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 4 & -1 \\ 1 & 5 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 5 & -4 \\ 3 & 4 & -1 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 - 5C\ ,\ C_3 + 4C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & -11 & -11 \\ 2 & -11 & -11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 5 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & -11 & -11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 - R_1$, $R_3 - 2R_1$,

$$\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & -11 & 11\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\-1 & 1 & -1\\1 & 0 & -2\end{bmatrix} A \begin{bmatrix}1 & 5 & -4\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$C_3 + C_2$

$$\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & -11 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\-1 & 1 & -1\\1 & 0 & -2\end{bmatrix} A \begin{bmatrix}1 & -5 & -1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}$$

$R_3 \times \frac{1}{11}$,

$$\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\-1 & 1 & -1\\1/11 & 0 & -2/11\end{bmatrix} A \begin{bmatrix}1 & -5 & -1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\-1/11 & 0 & 2/11\\-1 & 1 & -1\end{bmatrix} A \begin{bmatrix}1 & -5 & -1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}$$

Thus

$$P = \begin{bmatrix}0 & 0 & 1\\-1/11 & 0 & 2/11\\-1 & 1 & -1\end{bmatrix} \quad \Delta \quad |P| = \frac{-1}{11}$$

$$Q = \begin{bmatrix}1 & -5 & -1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix} \quad \Delta \quad |Q| = 1$$

P and Q are non-singular matrices
Also Rank of A = 2

ii) $$A = \begin{bmatrix}2 & 1 & -3 & -6\\3 & -3 & 1 & 2\\1 & 1 & 1 & 2\end{bmatrix}$$

Solutions:

Consider

$$A = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

$R_2 - 6R_3$,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 28 & 56 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 6 & 1 & 9 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 - 2C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 28 & 0 \\ 0 & -1 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -6 & 1 & 9 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 - 5C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 28 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -6 & 1 & 9 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \times \frac{1}{28}$, $R_3 \times (-1)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 3/14 & 1/28 & 9/28 \\ -1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ 3/14 & 1/28 & 9/28 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[I_3 \quad 0] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 3/14 & 1/28 & 9/28 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore\ P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 3/14 & 1/28 & 9/28 \end{bmatrix}, \ |P| = \frac{1}{28}$$

$$Q = \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}, |Q| = 1$$

$\therefore$ P&Q are non singular.

Also,

Rank of A = 3.

SYSTEM OF LINEAR EQUATIONS:

i) Consider a set of equations :

$$a_1x+b_1y+c_1z=d_1$$

$$a_2x+b_2y+c_2z=d_2$$

$$a_3x+b_3y+c_3z=d_3$$

The equation can be written in the matrix form as :

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

A X D

i.e. $AX=D$

Now we join matrices A and D

$$[A:D]=\begin{bmatrix} a_1 & b_1 & c_1 & : & d_1 \\ a_2 & b_2 & c_2 & : & d_2 \\ a_3 & b_3 & c_3 & : & d_3 \end{bmatrix}$$

It is called as Augment matrix

We reduce (A.D.) to Echelon form and thereby find the ranks of A and (A:D)

1) If $\rho(A)\neq\rho(AD)$ then the system is inconsistent i.e. it has no solution.

2) If $\rho(AD)=\rho(A)$ then the system is consistent and if

(i) $\rho(AD)=\rho(A)$ =Number of unknowns then the system is consistent and has unique solution.

(ii) $\rho(AD)=\rho(A)<$ Number of unknowns and has infinitely many solutions.

Non- Homogeneous equation:-

System of simultaneous equation in the matrix form is AX=D…..(I)

Pre-multiplying both sides of I by A^{-1} we set

$\therefore A^{-1}AX = A^{-1}D$
$\therefore IX = A^{-1}B$
$\therefore X = A^{-1}B$
which is required solution of the given non-homogeneous equation.

Homogeneous linear equation:-

Consider the system of simultaneous equations in the matrix form.
$AX = D$
If all elements of D are zero

i.e

then the system of equation is known as homogeneous system of equations.

In this case coefficient matrix A and the augmented matrix [A,O] are the same. So The rank is same. It follow that the system has solution

$x_1, x_2, x_3 \ldots\ldots x_4 = 0$, which is called a trivial solution.

Example 2: Solve the following system of equations

$$2x_1 - 3x_2 + x_3 = 0$$
$$x_1 + 2x_2 - 3x_3 = 0$$
$$4x_1 - x_2 - 2x_3 = 0$$

Solution: The system is written as

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & -3 \\ 4 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence the coefficient and augmented matrix are the same
We consider

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & -3 \\ 4 & -1 & -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & -3 \\ 4 & -1 & -2 \end{bmatrix}$$

$R_1 \Rightarrow R_1 \leftrightarrow R_2$

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ 4 & -1 & -2 \end{bmatrix}$$

$R_2 \Rightarrow R_2 - 2R_1$ & $R_3 \Rightarrow R_3 - 4R_1$

$$= \begin{bmatrix} 1 & 2 & -3 \\ 0 & -7 & 7 \\ 0 & -9 & -10 \end{bmatrix}$$

$R_2 \Rightarrow R_2 \times \frac{1}{7}$

$$= \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & -9 & -10 \end{bmatrix}$$

$R_3 \Rightarrow R_3 + 9R_2$ & $R_1 \Rightarrow R_1 - 2R_2$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -19 \end{bmatrix}$$

$R_3 \Rightarrow R_3 \times -\frac{1}{7}$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \Rightarrow R_2 + R_3$ & $R_1 \Rightarrow R_1 + R_3$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence Rank of A is 3

$\therefore \ell(A) = 3,$

The coefficient matrix is non-singular

Therefore there exist a trivial solution

$x_1 = x_2 = x_3 = 0$

Example 3: Solve the following system of equations

$$x_1 + 3x_2 - 2x_3 = 0$$
$$2x_1 - x_2 + 4x_3 = 0$$
$$x_1 - 11x_2 + 14x_3 = 0$$

Solution: The given equations can be written as

$$AX = 0$$

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & 11 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here the coefficient & augmented matrix are the same

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$$

$R_2 \Rightarrow R_2\text{-}2\,R_1 \;\&\; R_3 \Rightarrow R_3 - R_1$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$$

$R_3 \Rightarrow R_3 - 2R_2$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

Here rank of A is 2 i.e

$$\ell(A) = 2$$

So the system has infinite non-trivial solutions.

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & -8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 3x_2 - 2x_3 = 0$

$-7x_2 - 8x_3 = 0$

$7x_2 = 8x_3$

$x_2 = \frac{8}{7}x_3$

Let $x_3 - 8x_3 = \lambda$

$\therefore x_2 = \frac{8}{7}\lambda$

$\therefore x_1 + 3\left(\frac{8}{7}\lambda\right) - 2\lambda = 0$

$\therefore x_1 + \frac{24}{7}\lambda - 2\lambda = 0$

$\therefore x_1 = 2\lambda - \frac{24}{7}\lambda$

$\therefore x_1 = -\frac{10}{7}\lambda$

Hence $x_1 = -\frac{10}{7}\lambda$ $x_2 = \frac{8}{7}\lambda$ and $x_3 = \lambda$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{10}{7}\lambda \\ \frac{8}{7}\lambda \\ \lambda \end{bmatrix}$$

Hence infinite solution as deferred upon value of λ

PROBLEMS

Discuss the consistency of

$x+3y-4z=-2$

$-y+3z=4$

$x+2y-z=-5$

olution: In the matrix form

$$\begin{bmatrix} 2 & 3 & -4 \\ 1 & -1 & 3 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -5 \end{bmatrix}$$

onsider an Agumental matrix

$$[A:D]=\begin{bmatrix} 2 & 3 & -4 & : & -2 \\ 1 & -1 & 3 & : & 4 \\ 3 & 2 & -1 & : & -5 \end{bmatrix}$$

$$R_2 \to R_2 - \frac{1}{2} R_1$$

$$R_3 \to R_3 - \frac{3}{2} R_1$$

$$[A:D]=\begin{bmatrix} 2 & 3 & -4 & : & -2 \\ 0 & -5/2 & 5 & : & 5 \\ 0 & -5/2 & 5 & : & -2 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$= \begin{bmatrix} 2 & 3 & -4 & -2 \\ 0 & \frac{-5}{2} & 5 & 5 \\ 0 & 0 & 0 & -7 \end{bmatrix}$$

$$\therefore \ \rho(AD)=3$$

$$\rho(A)=2$$

$$\therefore \ \rho(AD) \neq \rho(A)$$

$\therefore$ The system is inconsistent and it has no solution.

Discuss the consistency of

Example 5: Discuss the consistency of

$3x+y+2z=3$
$2x-3y-z=-3$
$x+2y+z=4$

Solution: In the matrix form,

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

$$A \qquad X = D$$

Now we join matrices A and D

Consider

$$[A:D] = \begin{bmatrix} 3 & 1 & 2 & : & 3 \\ 2 & -3 & -1 & : & -3 \\ 1 & 2 & 1 & : & 4 \end{bmatrix}$$

We reduce to Echelon form

$R_1 \to R_3$

$$[A:D] = \begin{bmatrix} 1 & 2 & 1 & : & 4 \\ 2 & -3 & -1 & : & -3 \\ 3 & 1 & 2 & : & 3 \end{bmatrix}$$

We reduce to Echelon form

$R_1 \to R_3$

$$[A:D] = \begin{bmatrix} 1 & 2 & 1 & : & 4 \\ 2 & -3 & -1 & : & -3 \\ 3 & 1 & 2 & : & 3 \end{bmatrix}$$

$R_2 \to R_2 - 2\,R_1$

$R_3 \to R_3 - 3\,R_1$

$$[A:D] = \begin{bmatrix} 1 & 2 & 1 & : & 4 \\ 2 & -7 & -3 & : & -11 \\ 0 & -5 & -1 & : & -9 \end{bmatrix}$$

$R_3 \to R_3 - \frac{5}{7}\,R_2$

$$[A:D] = \begin{bmatrix} 1 & 2 & 1 & : & 4 \\ 0 & -7 & -3 & : & -11 \\ 0 & 0 & 8/7 & : & -8/7 \end{bmatrix} \text{.........(1)}$$

This is in Echelon form

$\therefore \rho(AD) = 3$

$\rho(A) = 3$

$\therefore$ ρ (AD) = ρ (A) = Number of unknowns

$\therefore$ system is consist and has unique solution.

Step (2) : To find the solution we proceed as follows. At the end of the row transformation the value of z is calculated then values of y and the value of x in the last.

The matrix in e.g. (1) in Echelon form can be written as

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & -3 \\ 0 & 0 & 8/7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ -8/7 \end{bmatrix}$$

$\therefore$ Expanding by R_3

$\frac{8}{7} z = -8/7$

$\therefore$ $z = -1$

$$-7y-3z=-11$$
$$-7y-3(-1)=-11$$
$$-7y+3=-11$$
$$\cancel{-}7y=\cancel{-}14$$
$$y=2$$
exp *anding* by R_1
$$x+2y+z=4$$
$$x+4-1=4$$
$$\therefore\ x=1$$
$$\therefore\ x=1,\ y=2, z=-1$$

Examine for consistency and solve

$$5x+3y+7z=4$$
$$3x+26y+2z=9$$

$$7x+2y+10z=5$$

Solution:

Step (1) : In the matrix form

$$\begin{bmatrix}5 & 3 & 7\\ 3 & 26 & 2\\ 7 & 2 & 10\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}4\\ 5\\ 6\end{bmatrix}$$
$$A \qquad X = D$$

Consider

$$[A:D]=\begin{bmatrix}5 & 3 & 7 & : & 4\\ 3 & 26 & 2 & : & 9\\ 7 & 2 & 10 & : & 5\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}4\\ 5\\ 6\end{bmatrix}$$

$$R_1 \to \frac{1}{5} R_1$$

$$[A:D]=\begin{bmatrix}1 & 3/5 & 7/5 & : & 4/5\\ 3 & 26 & 2 & : & 9\\ 7 & 2 & 10 & : & 5\end{bmatrix}$$

$$R_2 \to R_2-3R_1$$
$$R_3 \to R_3-7R_1$$

$$[A:D]=\begin{bmatrix}1 & 3/5 & 7/5 & : & 4/5\\ 0 & 121/5 & -11/5 & : & 33/5\\ 0 & -11/5 & 1/5 & : & -3/5\end{bmatrix}$$

$$\therefore\quad \rho\ (AD)=2$$
$$\rho(A)=2$$
$$\therefore\quad \rho(AD)=\rho(A)=2<3=\textit{Number} \text{ of unknowns}$$

The system is consistent and has infinitely many solutions.

Step (2) :- To find the solution we proceed as follows:

Let

$\therefore$ By expanding R_2

$121/5y - 11/5z = 33/5$

$\therefore$ $11y - z = 3$

$\therefore$ $y = \dfrac{z+3}{11}$

$\therefore$ put $z = k$

$\therefore$ $y = \dfrac{k+3}{11}$

By exapanding R_1

$x + \frac{3}{5}y + \frac{7}{5}z = \frac{4}{5}$

$\therefore$ $x = \dfrac{7}{11} - \dfrac{16k}{11}$

Working rule to find characteristic equation:

For a 2 x 2 matrix:

Method 1:

The characteristic equation is $|A - \lambda I| = 0$

Method 2:

Its characteristic equation can be written as $\lambda^2 - S_1\lambda + S_2 = 0$ where $S_1 = sum\ of\ the\ main\ diagonal\ elements$, $S_2 = Determinant\ of\ A = |A|$

1. Find the characteristic equation of the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

Solution: Let A = $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$. Its characteristic equation is $\lambda^2 - S_1\lambda + S_2 = 0$where $S_1 = sumofthemaindiagonalelements = 1 + 2 = 3$,

$S_2 = DeterminantofA = |A|$ = 1(2) – 2(0) = 2

Therefore, the characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$

For a 3 x 3 matrix:

Method 1:

The characteristic equation is $|A - \lambda I| = 0$

Method 2:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum\ of\ the\ main\ diagonal\ elements,$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements$,

$S_3 = DeterminantofA = |A|$

2. **Find the characteristic equation of** $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$

Solution: Its characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$, where

$S_1 = sum of the main diagonal elements$ = 8 + 7 + 3 = 18,

$S_2 = Sum of the minors of the main diagonal elements = \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix} = 5 + 20 + 20 = 45$, $S_3 = Determinant of A = |A|$ = 8(5)+6(-10)+2(10) = 40 -60 + 20 = 0

Therefore, the characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

Note:

1. Solving $|A - \lambda I| = 0$, we get n roots for λ and these roots are called characteristic roots or eigen values or latent values of the matrix A
2. Corresponding to each value of λ, the equation AX = λX has a non-zero solution vector X

   If X_r be the non-zero vector satisfying AX = λX, when $\lambda = \lambda_r$, X_r is said to be the latent vector or eigen vector of a matrix A corresponding to λ_r

CAYLEY-HAMILTON THEOREM:

Statement: Every square matrix satisfies its own characteristic equation

Uses of Cayley-Hamilton theorem:

(1) To calculate the positive integral powers of A
(2) To calculate the inverse of a square matrix A

1. **Show that the matrix** $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ **satisfies its own characteristic equation**

Solution:Let A = $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$. The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where $S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 1 = 2$

$S_2 = |A| = 1 - (-4) = 5$

The characteristic equation is $\lambda^2 - 2\lambda + 5 = 0$

To prove $A^2 - 2A + 5I = 0$

$A^2 = A(A) = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix}$

$A^2 - 2A + 5I = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Therefore, the given matrix satisfies its own characteristic equation

3. Verify Cayley-Hamilton theorem, find A^4 and A^{-1} when A = $\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Solution: The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 2 + 2 + 2 = 6$

$S_2 = Sum\ of\ the\ minirs\ of\ the\ main\ diagonal\ elements = 3 + 2 + 3 = 8$

$S_3 = \ |A| \ = 2(4-1) + 1(-2+1) + 2(1-2) = \ 2(3) - 1 - 2 = 3$

Therefore, the characteristic equation is $\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$

To prove that: $A^3 - 6A^2 + 8A - 3I = 0$------------- (1)

$$A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = \ A^2(A) = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

$A^3 - 6A^2 + 8A - 3I$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - \begin{bmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{bmatrix} + \begin{bmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

To find A^4:

(1) $\Rightarrow$ $A^3 - 6A^2 + 8A - 3I = 0 \Rightarrow A^3 = 6A^2 - 8A + 3I$ ------------- (2)

Multiply by A on both sides, $A^4 = 6A^3 - 8A^2 + 3A = 6(6A^2 - 8A + 3I) - \ 8A^2 + 3A$

Therefore, $A^4 = 36A^2 - 48A + 18I - 8A^2 + 3A = 28A^2 - 45A + 18I$

Hence, $A^4 = 28\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 45\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 18\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 196 & -168 & 252 \\ -140 & 168 & -168 \\ 140 & -140 & 196 \end{bmatrix} - \begin{bmatrix} 90 & -45 & 90 \\ -45 & 90 & -45 \\ 45 & -45 & 90 \end{bmatrix} + \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix} =$$

$$\begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$$

To find A^{-1}:

Multiplying (1) by $A^{-1}, A^2 - 6A + 8I - 3A^{-1} = 0$

$\Rightarrow 3A^{-1} = A^2 - 6A + 8I$

$$\Rightarrow 3A^{-1} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - \begin{bmatrix} -12 & 6 & -12 \\ 6 & -12 & 6 \\ -6 & 6 & -12 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

5. Find A^{-1} if A = $\begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$, using Cayley-Hamilton theorem

Solution:The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 2 - 1 = 2$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements = (-2+1) + (-1-8) + (2+3)$
$= -1 - 9 + 5 = -5$

$S_3 = |A| = 1(-2+1) + 1(-3+2) + 4(3-4) = -1 - 1 - 4 = -6$

The characteristic equation of A is $\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

By Cayley- Hamilton theorem, $A^3 - 2A^2 - 5A + 6I = 0$ ------------ (1)

To find A^{-1}:

Multiplying (1) by A^{-1}, we get, $A^2 - 2A - 5A^{-1}A + 6A^{-1}I = 0 \Rightarrow A^2 - 2A - 5I + 6A^{-1} = 0$

$6A^{-1} = -A^2 + 2A + 5I \Rightarrow A^{-1} = \frac{1}{6}(-A^2 + 2A + 5I)$ ------------- (2)

$$A^2 = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1-3+8 & -1-2+4 & 4+1-4 \\ 3+6-2 & -3+4-1 & 12-2+1 \\ 2+3-2 & -2+2-1 & 8-1+1 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 1 \\ 7 & 0 & 11 \\ 3 & -1 & 8 \end{bmatrix}$$

$$-A^2 + 2A + 5I = \begin{bmatrix} -6 & -1 & -1 \\ -7 & 0 & -11 \\ -3 & 1 & -8 \end{bmatrix} + \begin{bmatrix} 2 & -2 & 8 \\ 6 & 4 & -2 \\ 4 & 2 & -2 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{bmatrix}$$

From (2), $A^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{bmatrix}$

EIGEN VALUES AND EIGEN VECTORS OF A REAL MATRIX:

Working rule to find eigen values and eigen vectors:

1. Find the characteristic equation $|A - \lambda I| = 0$
2. Solve the characteristic equation to get characteristic roots. They are called eigen values
3. To find the eigen vectors, solve $[A - \lambda I]X = 0$ for different values of λ

Note:

1. Corresponding to n distinct eigen values, we get n independent eigen vectors
2. If 2 or more eigen values are equal, it may or may not be possible to get linearly independent eigen vectors corresponding to the repeated eigen values
3. If X_i is a solution for an eigen value λ_i, then cX_i is also a solution, where c is an arbitrary constant. Thus, the eigen vector corresponding to an eigen value is not unique but may be any one of the vectors cX_i
4. Algebraic multiplicity of an eigen value λ is the order of the eigen value as a root of the characteristic polynomial (i.e., if λ is a double root, then algebraic multiplicity is 2)
5. Geometric multiplicity of λ is the number of linearly independent eigen vectors corresponding to λ

Non-symmetric matrix:

If a square matrix A is non-symmetric, then A ≠ A^T

Note:

1. In a non-symmetric matrix, if the eigen values are non-repeated then we get a linearly independent set of eigen vectors
2. In a non-symmetric matrix, if the eigen values are repeated, then it may or may not be possible to get linearly independent eigen vectors.
   If we form a linearly independent set of eigen vectors, then diagonalization is possible through similarity transformation

Symmetric matrix:

If a square matrix A is symmetric, then A = A^T

Note:

1. In a symmetric matrix, if the eigen values are non-repeated, then we get a linearly independent and pair wise orthogonal set of eigen vectors
2. In a symmetric matrix, if the eigen values are repeated, then it may or may not be possible to get linearly independent and pair wise orthogonal set of eigen vectors
   If we form a linearly independent and pair wise orthogonal set of eigen vectors, then diagonalization is possible through orthogonal transformation

1. Find the eigen values and eigen vectors of the matrix $\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$

<u>Solution</u>: Let A = $\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$ which is a non-symmetric matrix

<u>To find the characteristic equation:</u>

The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where
$S_1 = sum of the main diagonal elements = 1 - 1 = 0,$

$S_2 = Determinant of A =$ $|A|$ = 1(-1) − 1(3) = - 4

Therefore, the characteristic equation is $\lambda^2 - 4 = 0$ i.e., $\lambda^2 = 4$ or $\lambda = \pm 2$

Therefore, the eigen values are 2, -2

A is a non-symmetric matrix with non- repeated eigen values

<u>To find the eigen vectors:</u>

$$[A - \lambda I]X = 0$$

$$\left[\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$--------------- **(1)**

Case 1: If $\lambda = -2$, $\begin{bmatrix} 1-(-2) & 1 \\ 3 & -1-(-2) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ **[From (1)]**

i.e., $\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

i.e., $3x_1 + x_2 = 0$

$$3x_1 + x_2 = 0$$

i.e., we get only one equation $3x_1 + x_2 = 0 \Rightarrow 3x_1 = -x_2 \Rightarrow \frac{x_1}{1} = \frac{x_2}{-3}$

Therefore $X_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

Case 2: If $\lambda = 2$, $\begin{bmatrix} 1-(2) & 1 \\ 3 & -1-(2) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ **[From (1)]**

i.e., $\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

i.e., $-x_1 + x_2 = 0 \Rightarrow x_1 - x_2 = 0$

$$3x_1 - 3x_2 = 0 \Rightarrow x_1 - x_2 = 0$$

i.e., we get only one equation $x_1 - x_2 = 0$

$$\Rightarrow x_1 = x_2 \Rightarrow \frac{x_1}{1} = \frac{x_2}{1}$$

Hence, $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2. **Find the eigen values and eigen vectors of $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$**

Solution: Let A = $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ which is a non-symmetric matrix

To find the characteristic equation:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum of the main diagonal elements = 2+3+2 = 7,$

$S_2 = Sum of the minors of the main diagonal elements = \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4+3+4 = 11,$

$S_3 = Determinant of A = |A|$ = 2(4)-2(1)+1(-1) = 5

Therefore, the characteristic equation of A is $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

1	1	-7	11	-5
	0	1	-6	5
	1	-6	5	0

$(\lambda - 1)(\lambda^2 - 6\lambda + 5) = 0 \Rightarrow \lambda = 1,$

$$\lambda = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(5)}}{2(1)} = \frac{6 \pm \sqrt{16}}{2} = \frac{6 \pm 4}{2} = \frac{6+4}{2}, \frac{6-4}{2} = 5, 1$$

Therefore, the eigen values are 1, 1, and 5

A is a non-symmetric matrix with repeated eigen values

To find the eigen vectors:

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\lambda = 5$, $\begin{bmatrix} 2-5 & 2 & 1 \\ 1 & 3-5 & 1 \\ 1 & 2 & 2-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -3x_1 + 2x_2 + x_3 = 0$ --------- (1)

$x_1 - 2x_2 + x_3 = 0$ ------------- (2)

$x_1 + 2x_2 - 3x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$x_1 x_2 x_3$$

$$\begin{array}{cccc} 2 & 1 & -3 & 2 \\ -2 & 1 & 1 & -2 \end{array}$$

$$\Rightarrow \frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Therefore, $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Case 2: If $\lambda = 1$, $\begin{bmatrix} 2-1 & 2 & 1 \\ 1 & 3-1 & 1 \\ 1 & 2 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow x_1 + 2x_2 + x_3 = 0$$

$$x_1 + 2x_2 + x_3 = 0$$

$$x_1 + 2x_2 + x_3 = 0$$

All the three equations are one and the same. Therefore, $x_1 + 2x_2 + x_3 = 0$

Put $x_1 = 0 \Rightarrow 2x_2 + x_3 = 0 \Rightarrow 2x_2 = -x_3 . Taking x_3 = 2 , x_2 = -1$

Therefore, $X_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

Put $x_2 = 0 \Rightarrow x_1 + x_3 = 0 \Rightarrow x_3 = -x_1. Taking x_1 = 1, x_3 = -1$

Therefore, $X_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

4. Find the eigen values and eigen vectors of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Solution: Let A $= \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ which is a symmetric matrix

To find the characteristic equation:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sumofthemaindiagonalelements = 1 + 5 + 1 = 7,$

$S_2 = Sumoftheminorsofthemaindiagonalelements = \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} = 4 - 8 + 4 = 0,$

$S_3 = Determinantof A =$ $|A|$ = 1(4)-1(-2)+3(-14) = - 4 + 2-42 = - 36

Therefore, the characteristic equation of A is $\lambda^3 - 7\lambda^2 + 0\lambda - 36 = 0$

$$\begin{array}{c|cccc} -2 & 1 & -7 & 0 & 36 \\ & 0 & -2 & 18 & -36 \\ \hline & 1 & -9 & 18 & \boxed{0} \end{array}$$

$(\lambda - (-2))(\lambda^2 - 9\lambda + 18) = 0 \Rightarrow \lambda = -2,$

$$\lambda = \frac{9 \pm \sqrt{(-9)^2 - 4(1)(18)}}{2(1)} = \frac{9 \pm \sqrt{81 - 72}}{2} = \frac{9 \pm 3}{2} = \frac{9+3}{2}, \frac{9-3}{2} = 6, 3$$

Therefore, the eigen values are -2, 3, and 6

A is a symmetric matrix with non- repeated eigen values

To find the eigen vectors:

$[A - \lambda I]X = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\lambda = -2$, $\begin{bmatrix} 1-(-2) & 1 & 3 \\ 1 & 5-(-2) & 1 \\ 3 & 1 & 1-(-2) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 3x_1 + x_2 + 3x_3 = 0$ --------- (1)

$x_1 + 7x_2 + x_3 = 0$ ------------- (2)

$3x_1 + x_2 + 3x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$\begin{array}{ccccccc} & & x_1 & & x_2 & & x_3 \\ 1 & & 3 & & 3 & & 1 \\ 7 & & 1 & & 1 & & 7 \end{array}$$

$$\Rightarrow \frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20} \Rightarrow \frac{x_1}{-4} = \frac{x_2}{0} = \frac{x_3}{4} = \frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

Therefore, $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Case 2: If $\lambda = 3,$ $\begin{bmatrix} 1-3 & 1 & 3 \\ 1 & 5-3 & 1 \\ 3 & 1 & 1-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -2x_1 + x_2 + 3x_3 = 0$ ---------- (1)

$x_1 + 2x_2 + x_3 = 0$ ---------------- (2)

$3x_1 + x_2 - 2x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$\begin{array}{ccccccc} & & x_1 & & x_2 & & x_3 \\ 1 & & 3 & & -2 & & 1 \\ 2 & & 1 & & 1 & & 2 \end{array}$$

$$\Rightarrow \frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{-1} = \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

Therefore, $X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Case 3: If $\lambda = 6,$ $\begin{bmatrix} 1-6 & 1 & 3 \\ 1 & 5-6 & 1 \\ 3 & 1 & 1-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -5x_1 + x_2 + 3x_3 = 0$ ---------- (1)

$x_1 - x_2 + x_3 = 0$ ---------------- (2)

$3x_1 + x_2 - 5x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$x_1 x_2 x_3$$

$$\begin{array}{cccc} 1 & 3 & -5 & 1 \\ -1 & 1 & 1 & -1 \end{array}$$

$\Rightarrow \frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4} \Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$

Therefore, $X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

5. **Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Determine the algebraic and geometric multiplicity**

Solution: Let A $= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ which is a symmetric matrix

To find the characteristic equation:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum\ of\ the\ main\ diagonal\ elements = 0 + 0 + 0 = 0,$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} =$

$-1 - 1 - 1 = -3,$

$S_3 = Determinant\ of\ A = |A|$ = 0 -1(-1)+ 1(1) = 0 + 1 + 1 = 2

Therefore, the characteristic equation of A is $\lambda^3 - 0\lambda^2 - 3\lambda - 2 = 0$

$$\begin{array}{c|cccc} -1 & 1 & 0 & -3 & -2 \\ & 0 & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & 0 \end{array}$$

$(\lambda - (-1))(\lambda^2 - \lambda - 2) = 0 \Rightarrow \lambda = -1,$

$$\lambda = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = \frac{1+3}{2}, \frac{1-3}{2} = 2, -1$$

Therefore, the eigen values are 2, -1, and -1

A is a symmetric matrix with repeated eigen values. The algebraic multiplicity of $\lambda = -1$ is 2

To find the eigen vectors:

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 0-\lambda & 1 & 1 \\ 1 & 0-\lambda & 1 \\ 1 & 1 & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\lambda = 2$, $\begin{bmatrix} 0-2 & 1 & 1 \\ 1 & 0-2 & 1 \\ 1 & 1 & 0-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -2x_1 + x_2 + x_3 = 0$ --------- (1)

$x_1 - 2x_2 + x_3 = 0$ ------------- (2)

$x_1 + x_2 - 2x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$x_1 \quad x_2 \quad x_3$$

$$\begin{array}{cccc} 1 & 1 & -2 & 1 \\ -2 & 1 & 1 & -2 \end{array}$$

$\Rightarrow \frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$

Therefore, $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Case 2: If $\lambda = -1$, $\begin{bmatrix} 0-(-1) & 1 & 1 \\ 1 & 0-(-1) & 1 \\ 1 & 1 & 0-(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 + x_2 + x_3 = 0$ ---------- (1)

$x_1 + x_2 + x_3 = 0$ ---------------- (2)

$x_1 + x_2 + x_3 = 0$ ------------ (3). All the three equations are one and the same.

Therefore, $x_1 + x_2 + x_3 = 0$. Put $x_1 = 0 \Rightarrow x_2 + x_3 = 0 \Rightarrow x_3 = -x_2 \Rightarrow \frac{x_2}{1} = \frac{x_3}{-1}$

Therefore, $X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Since the given matrix is symmetric and the eigen values are repeated, let $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$. X_3 is orthogonal to X_1 and X_2 .

$[1 \quad 1 \quad 1] \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \Rightarrow l + m + n = 0$ ------------ (1)

$[0 \quad 1 \quad -1] \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \Rightarrow 0l + m - n = 0$-------- (2)

Solving (1) and (2) by method of cross-multiplication, we get,

$$\begin{array}{ccccccc} & l & & m & & n & \\ 1 & & 1 & & 1 & & 1 \\ 1 & & -1 & & 0 & & 1 \end{array}$$

$\frac{l}{-2} = \frac{m}{1} = \frac{n}{1}$. Therefore, $X_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

Thus, for the repeated eigen value $\lambda = -1$, there corresponds two linearly independent eigen vectors X_2 and X_3. So, the geometric multiplicity of eigen value $\lambda = -1$ is 2

Problems under properties of eigen values and eigen vectors.

1. **Find the sum and product of the eigen values of the matrix** $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

   Solution: Sum of the eigen values = Sum of the main diagonal elements = -3

   Product of the eigen values = $|A|$ = -1 (1 – 1) -1(-1 – 1) + 1(1- (-1)) = 2 + 2 = 4

2. **Product of two eigen values of the matrix A =** $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ **is 16. Find the third eigen value**

   Solution: Let the eigen values of the matrix be $\lambda_1, \lambda_2, \lambda_3$.

   Given $\lambda_1 \lambda_2 = 16$

We know that $\lambda_1\lambda_2\lambda_3 = |A|$ (Since product of the eigen values is equal to the determinant of the matrix)

$\lambda_1\lambda_2\lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$ = 6(9-1)+2(-6+2) +2(2-6) = 48-8-8 = 32

$$There\ fore, \lambda_1\lambda_2\lambda_3 = 32 \Rightarrow 16\lambda_3 = 32 \Rightarrow \lambda_3 = 2$$

3. **Find the sum and product of the eigen values of the matrix A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ without finding the roots of the characteristic equation**

   Solution:We know that the sum of the eigen values = Trace of A = a + d

   Product of the eigen values = |A| = ad – bc

4. **If 3 and 15 are the two eigen values of A = $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$, find |A|, without expanding the determinant**

Solution:Given $\lambda_1 = 3\ and\ \lambda_2 = 15, \lambda_3 =?$

We know that sum of the eigen values = Sum of the main diagonal elements

$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 8 + 7 + 3$

$$\Rightarrow 3 + 15 + \lambda_3 = 18 \Rightarrow \lambda_3 = 0$$

$We\ know\ that\ the\ product\ of\ the\ eigen\ values = |A|$

$\Rightarrow (3)(15)(0) = |A|$
$\Rightarrow |A| = 0$

5. **If 2, 2, 3 are the eigen values of A = $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$, find the eigen values of A^T**

Solution:By the property "A square matrix A and its transpose A^T have the same eigen values", the eigen values of $A^T\ are\ 2,2,3$

6. **Find the eigen values of A = $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 4 & 4 \end{bmatrix}$**

Solution:Given A = $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 4 & 4 \end{bmatrix}$ .Clearly, A is a lower triangular matrix. Hence, by the property "the characteristic roots of a triangular matrix are just the diagonal elements of the matrix", the eigen values of A are 2, 3, 4

7. **Two of the eigen values of A = $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ are 3 and 6. Find the eigen values of A^{-1}**

Solution:Sum of the eigen values = Sum of the main diagonal elements = 3 +5+3 = 11

Given 3,6 are two eigen values of A. Let the third eigen value be k.

Then, 3 + 6 + k = 11 $\Rightarrow k = 2$

Therefore, the eigen values of A are 3, 6, 2

By the property "If the eigen values of A are$\lambda_1, \lambda_2, \lambda_3$, then the eigen values of A^{-1} are$\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$", the eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$

8. Find the eigen values of the matrix$\begin{bmatrix} \mathbf{1} & \mathbf{-2} \\ \mathbf{-5} & \mathbf{4} \end{bmatrix}$. Hence, form the matrix whose eigen values are $\frac{1}{6}$ $\boldsymbol{and - 1}$

<u>**Solution**</u>: Let A =$\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$. The characteristic equation of the given matrix is $\lambda^2 - S_1\lambda + S_2 = 0$ where $S_1 = Sum\ of\ the\ main\ diagonal\ elements =\ 5$ and $S_2 =\ |A| = -6$

$S_2 = 0$ where $S_1 = Sum\ of\ the\ main\ diagonal\ elements =\ 5$ and $S_2 =\ |A| = -6$

Therefore, the characteristic equation is $\lambda^2 - 5\lambda - 6 = 0 \Rightarrow \lambda = \frac{5 \pm \sqrt{(-5)^2 - 4(1)(-6)}}{2(1)} = \frac{5 \pm 7}{2} = 6, -1$

Therefore, the eigen values of A are 6, -1

Hence, the matrix whose eigen values are $\frac{1}{6}$ $and - 1$ is A^{-1}

$$A^{-1} = \frac{1}{|A|}\ adj\ A$$

$|$A$|$ = 4 -10 = - 6; adj A = $\begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix}$

Therefore, $A^{-1} = \frac{1}{-6}\begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix}$

9. Find the eigen values of the inverse of the matrix A = $\begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} & \mathbf{4} \\ \mathbf{0} & \mathbf{0} & \mathbf{4} \end{bmatrix}$

<u>**Solution**</u>:We know that A is an upper triangular matrix. Therefore, the eigen values of A are 2, 3, 4. Hence, by using the property "If the eigen values of A are$\lambda_1, \lambda_2, \lambda_3$, then the eigen values of A^{-1} are$\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$", the eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

10. Find the eigen values of A^3 given A = $\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{0} & \mathbf{2} & \mathbf{-7} \\ \mathbf{0} & \mathbf{0} & \mathbf{3} \end{bmatrix}$

<u>**Solution**</u>:Given A = $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -7 \\ 0 & 0 & 3 \end{bmatrix}$. A is an upper triangular matrix. Hence, the eigen values of A are 1, 2, 3

Therefore, the eigen values of A^3 are $1^3, 2^3, 3^3$ i.e., 1,8,27

11. If 1 and 2 are the eigen values of a 2 x 2 matrix A, what are the eigen values of A^2 $\boldsymbol{and}$ A^{-1}?

<u>**Solution**</u>:Given 1 and 2 are the eigen values of A.

Therefore, 1^2 and 2^2 i.e., 1 and 4 are the eigen values of A^2 and 1 and $\frac{1}{2}$ are the eigen values of A^{-1}

12. If 1,1,5 are the eigen values of A = $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$, find the eigen values of 5A

Solution:By the property "If $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of A, then $k\lambda_1, k\lambda_2, k\lambda_3$ are the eigen values of kA, the eigen values of 5A are 5(1), 5(1), 5(5) ie., 5,5,25

13. Find the eigen values of A, $A^2, A^3, A^4, 3A, A^{-1}, A - I, 3A^3 + 5A^2 - 6A + 2I$ if A = $\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$

Solution:Given A = $\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$. A is an upper triangular matrix. Hence, the eigen values of A are 2, 5

The eigen values of A^2 are $2^2, 5^2$ i.e., 4, 25

The eigen values of A^3 are $2^3, 5^3$ i.e., 8, 125

The eigen values of A^4 are $2^4, 5^4$ i.e., 16, 625

The eigen values of 3A are 3(2), 3(5) i.e., 6, 15

The eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{5}$

A – I = $\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$ - $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ = $\begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$

Since A - I is an upper triangular matrix, the eigen values of A- I are its main diagonal elements i.e., 1,4

Eigen values of $3A^3 + 5A^2 - 6A + 2I$ are $3\lambda_1^3 + 5\lambda_1^2 - 6\lambda_1 + 2$ and $3\lambda_2^3 + 5\lambda_2^2 - 6\lambda_2 + 2$ where $\lambda_1 = 2$ and $\lambda_2 = 5$

First eigen value = $3\lambda_1^3 + 5\lambda_1^2 - 6\lambda_1 + 2$

= $3(2)^3$+ $5(2)^2$ - 6(2) + 2 = 24 + 20 -12 + 2=34

Second eigen value = $3\lambda_2^3 + 5\lambda_2^2 - 6\lambda_2 + 2$

= $3(5)^3$+ $5(5)^2$- 6(5) + 2

= 375+ 125-30 + 2 = 472

14. Find the eigen values of adj A if A = $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Solution:Given A = $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. A is an upper triangular matrix. Hence, the eigen values of A are 3, 4, 1

We know that $A^{-1} = \frac{1}{|A|}\ adj\ A$

Adj A = |A| A^{-1}

The eigen values of A^{-1} are $\frac{1}{3}, \frac{1}{4}, 1$

|A| =Product of the eigen values = 12

Therefore, the eigen values of adj A is equal to the eigen values of 12 A^{-1} i.e., $\frac{12}{3}, \frac{12}{4}, 12$ i.e., 4, 3, 12

<u>Note:</u> A = $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$. Here, A is an upper triangular matrix, B is a lower triangular matrix and C is a diagonal matrix. In all the cases, the elements in the main diagonal are the eigen values. Hence, the eigen values of A, B and C are 1, 4, 6

15. Two eigen values of A = $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are equal and they are $\frac{1}{5}$ times the third. Find them

<u>Solution</u>:Let the third eigen value be λ_3

We know that $\lambda_1 + \lambda_2 + \lambda_3$ =2+3+2 = 7

Given $\lambda_1 = \lambda_2 = \frac{\lambda_3}{5}$

$$\frac{\lambda_3}{5} + \frac{\lambda_3}{5} + \lambda_3 = 7$$

$$\left[\frac{1}{5} + \frac{1}{5} + 1\right]\lambda_3 = 7 \Rightarrow \frac{7}{5}\lambda_3 = 7 \Rightarrow \lambda_3 = 5$$

Therefore, $\lambda_1 = \lambda_2 = 1$ and hence the eigen values of A are 1,1, 5

16. If 2, 3 are the eigen values of $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{bmatrix}$, *find the value of a*

<u>Solution</u>:Let A = $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{bmatrix}$. Let the eigen values of A be 2, 3, k

We know that the sum of the eigen values = sum of the main diagonal elements

Therefore, 2 +3 +k = 2+ 2+2 = 6 $\Rightarrow k = 1$

We know that product of the eigen values = $|A|$

$\Rightarrow$2(3)(k) = $|A|$

$$\Rightarrow 6 = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{vmatrix} \Rightarrow 6 = 2(4) - 0 + 1(-2a) \Rightarrow 6 = 8 - 2a \Rightarrow 2a = 2 \Rightarrow a = 1$$

17. Prove that the eigen vectors of the real symmetric matrix A = $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ are orthogonal in pairs

<u>Solution</u>:The characteristic equation of A is

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$where $S_1 = sum\ of\ the\ main\ diagonal\ elements = 7;$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements\ = 4 + (-8) + 4 = 0$

$$S_3 = |A| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{vmatrix} = 1(4) - 1(-2) + 3(-14) = -36$$

The characteristic equation of A is $\lambda^3 - 7\lambda^2 + 36 = 0$

3	1-7	0	36	
	0	3	-12	-36
	1	-4	-12	0

Therefore, $\lambda = 3, \lambda^2 - 4\lambda - 12 = 0 \Rightarrow \lambda = 3, \lambda = \frac{4\pm\sqrt{(-4)^2-4(1)(-12)}}{2(1)} = \frac{4\pm 8}{2} = 6, -2$

Therefore, the eigen values of A are -2, 3, 6

<u>To find the eigen vectors:</u>

$$(A - \lambda I)X = 0$$

<u>Case 1:</u> When $\lambda = -2, \begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$3x_1 + x_2 + 3x_3 = 0$ -------- (1)

$x_1 + 7x_2 + x_3 = 0$ ---------- (2)

$3x_1 + x_2 + 3x_3 = 0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication, we get,

$x_1 x_2 x_3$

$$\begin{matrix} 1 & & 3 & & 3 & & 1 \\ 7 & & 1 & & 1 & & 7 \end{matrix}$$

$$\frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20} \Rightarrow X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

<u>Case 2:</u>When $\lambda = 3, \begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-2x_1 + x_2 + 3x_3 = 0$ --------- (1)

$x_1 + 2x_2 + x_3 = 0$ --------- (2)

$3x_1 + x_2 - 2x_3 = 0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication, we get,

$x_1 x_2 x_3$

$$\begin{matrix} 1 & & 3 & & -2 & & 1 \\ 2 & & 1 & & 1 & & 2 \end{matrix}$$

$$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5} \Rightarrow X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

<u>Case 3:</u>When $\lambda = 6, \begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-5x_1 + x_2 + 3x_3 = 0$ --------- (1)

$x_1 - x_2 + x_3 = 0$ --------- (2)

$3x_1 + x_2 - 5x_3 = 0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication, we get,

$x_1 x_2 x_3$

$$\begin{array}{cccc} 1 & 3 & -5 & 1 \\ -1 & 1 & 1 & -1 \end{array}$$

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4} \Rightarrow X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Therefore, $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

To prove that: $X_1^T X_2 = 0, X_2^T X_3 = 0, X_3^T X_1 = 0$

$X_1^T X_2 = [-1 \quad 0 \quad 1] \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = -1 + 0 + 1 = 0$

$X_2^T X_3 = [1 - 1 \quad 1] \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 1 - 2 + 1 = 0$

$X_3^T X_1 = [1 \quad 2 \quad 1] \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -1 + 0 + 1 = 0$

Hence, the eigen vectors are orthogonal in pairs

18. Find the sum and product of all the eigen values of the matrix A = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 1 & 2 & 7 \end{bmatrix}$.Is the matrix singular?

<u>Solution</u>:Sum of the eigen values = Sum of the main diagonal elements =Trace of the matrix

Therefore, the sum of the eigen values = 1+2+7=10

Product of the eigen values = ∣A∣ = 1(14 - 8) -2(14 - 4) + 3(4 - 2) = 6-20+ 6= - 8

∣A∣≠0. Hence the matrix is non-singular.

19. Find the product of the eigen values of A = $\begin{bmatrix} 1 & 2 & -2 \\ 1 & 0 & 3 \\ -2 & -1 & -3 \end{bmatrix}$

<u>Solution</u>:Product of the eigen values of A = ∣A∣= $\begin{vmatrix} 1 & 2 & -2 \\ 1 & 0 & 3 \\ -2 & -1 & -3 \end{vmatrix} = 1(3) - 2(3) - 2(-1) =$

$3 - 6 + 2 = -1$

Theorem: The Eigen values of a Hermitian matrix are real.

Proof: Let A be Hermitian matrix. If X be the Eigen vector corresponding to the eigen value λ of A, then AX = λX -------------------- (1)

Pre multiplying both sides of (1) by X^{θ},we get

$X^{\theta}AX = \lambda X^{\theta}X$ ----------------------- (2)

Taking conjugate transpose of both sides of (2)

We get $\left(X^{\theta}AX\right)^{\theta} = \left(\lambda X^{\theta}X\right)^{\theta}$

i.e $X^{\theta}A^{\theta}\left(X^{\theta}\right)^{\theta} = \bar{\lambda}X^{\theta}\left(X^{\theta}\right)^{\theta}\left[\because (ABC)^{\theta} = C^{\theta}B^{\theta}A^{\theta} \; and \; (KA)^{\theta} = \bar{K}A^{\theta}\right]$

(or) $X^{\theta}A^{\theta}X = \bar{\lambda}X^{\theta}X\left[\because \left(X^{\theta}\right)^{\theta} = X, \left(A^{\theta}\right)^{\theta} = A\right]$-------------------- (3)

From (2) and (3), we have

$\lambda X^{\theta}X = \bar{\lambda}X^{\theta}X$

i.e $\left(\lambda - \bar{\lambda}\right)X^{\theta}X = 0 \Rightarrow \lambda - \bar{\lambda} = 0$

$\Rightarrow \lambda = \bar{\lambda}\left(\because X^{\theta}X \neq 0\right)$

$\therefore$ Hence λ is real.

Note: The Eigen values of a real symmetric are all real

Corollary: The Eigen values of a skew-Hermitian matrix are either purely imaginary (or) Zero

Proof: Let A be the skew-Hermitian matrix

If X be the Eigen vector corresponding to the Eigen value λ of A, then

$AX = \lambda X (or) (iA)X = (i\lambda)X$

From this it follows that $i\lambda$ is an Eigen value of iA

Which is Hermitian (since A is skew-hermitian)

$\therefore A^{\theta} = -A$

Now $(iA)^{\theta} = \bar{i}A^{\theta} = -iA^{\theta} = -i(-A) = iA$

Hence $i\lambda$ is real. Therefore λ must be either

Zero or purely imaginary.

Hence the Eigen values of skew-Hermitian matrix are purely imaginary or zero

Theorem 3: The Eigen values of an unitary matrix have absolute value l.

Proof: Let A be a square unitary matrix whose Eigen value is λ with corresponding eigen vector X

$\Rightarrow \quad AX = \lambda X \rightarrow (1)$

$\Rightarrow \quad \overline{A}\overline{X} = \bar{\lambda}\overline{X} \Rightarrow \overline{X}^{T}\overline{A}^{T} = \bar{\lambda}X^{T} \rightarrow (2)$

Since A is unitary, we have $\left(\overline{A}\right)^{T} A = I \rightarrow (3)$

(1) and (2) given $\overline{X}^{T}\overline{A}^{T} \; (AX) = \lambda\bar{\lambda}\overline{X}^{T}X$

i.e $\overline{X}^{T}X = \lambda\bar{\lambda}\overline{X}^{T}X$ From (3)

$\Rightarrow \overline{X}^T X\left(1-\lambda\overline{\lambda}\right)=0$

Since $\overline{X}^T X \neq 0$,we must have $1-\lambda\overline{\lambda}=0$

$\Rightarrow \lambda\overline{\lambda}=1$

Since $|\lambda|=|\overline{\lambda}|$

We must have $|\lambda|=1$

Theorem 4: Prove that transpose of a unitary matrix is unitary.

Proof: Let A be a unitary matrix

Then $A.A^{\theta}=A^{\theta}.A=I$

Where A^{θ} is the transposed conjugate of A.

$\therefore \left(AA^{\theta}\right)^T=\left(A^{\theta}A\right)^T=\left(I\right)^T$

$\therefore \left(AA^{\theta}\right)^T=\left(A^{\theta}A\right)^T=\left(I\right)^T$

$\Rightarrow \left(A^{\theta}\right)^T A^T=A^T\left(A^{\theta}\right)^T=I$

$\Rightarrow \left(A^T\right)^{\theta} A^T=A^T\left(A^T\right)^{\theta}=I$

Hence A^T is a unitary matrix.

PROBLEMS

1) Find the eigen values of A=$\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$

Sol: we have A= $\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$

So $\overline{A}=\begin{bmatrix} -3i & 2-i \\ -2-i & i \end{bmatrix}$ and $A^T=\begin{bmatrix} 3i & -2+i \\ 2+i & -i \end{bmatrix}$

$\Rightarrow \overline{A}=-A^T$

Thus A is a skew-Hermitian matrix.

$\therefore$ The characteristic equation of A is $|A-\lambda I|=0$

$\Rightarrow A^T=\begin{vmatrix} 3i-\lambda & -2+i \\ -2+i & -i-\lambda \end{vmatrix}=0$

$\Rightarrow \lambda^2-2i\lambda+8=0$

$\Rightarrow \lambda=4i,-2i$ are the Eigen values of A

2) Find the eigen values of $A = \begin{bmatrix} \frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}i \end{bmatrix}$

$$Now\ \overline{A} = \begin{bmatrix} -\frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}i \end{bmatrix} and$$

$$(\overline{A})^T = \begin{bmatrix} -\frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}i \end{bmatrix}$$

We can see that $\overline{A}^T.A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Thus A is a unitary matrix

$\therefore$ The characterstic equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} \frac{1}{2}i - \lambda & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}i - \lambda \end{vmatrix} = 0$$

Which gives $\lambda = \frac{\sqrt{3}}{2} + i\frac{1}{2}\ and\ \frac{-\sqrt{3}}{2} + \frac{1}{2}i$ and

$$\lambda = 1/2\sqrt{3} + 1/2i$$

Hence above λ values are Eigen values of A.

3) If A= $\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ **then show that**

A is Hermitian and iA is skew-Hermitian.

Sol: Given A= $\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ then

$\overline{A} = \begin{bmatrix} 3 & 7+4i & -2-5i \\ 7-4i & -2 & 3-i \\ -2+5i & 3+i & 4 \end{bmatrix}$ And $\overline{(A)}^T = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$

$\therefore A = (\overline{A})^T$ Hence A is Hermitian matrix.

Let B= iA

i.e B= $\begin{bmatrix} 3i & 4+7i & -5-2i \\ -4+7i & -2i & -1+3i \\ 5-2i & 1+3i & 4i \end{bmatrix}$ then

$$\overline{B} = \begin{bmatrix} -3i & 4-7i & -5+2i \\ -4-7i & 2i & -1-3i \\ 5+2i & 1-3i & -4i \end{bmatrix}$$

$$\left(\overline{B}\right)^T = \begin{bmatrix} -3i & -4-7i & 5+2i \\ 4-7i & 2i & 1-3i \\ -5+2i & -1-3i & -4i \end{bmatrix} = (-1)\begin{bmatrix} 3i & 4+7i & -5-2i \\ -4+7i & -2i & -1+3i \\ 5-2i & 1+3i & 4i \end{bmatrix} = -B$$

$\therefore \left(\overline{B}\right)^T$ =-B

$\therefore$ B= iA is a skew Hermitian matrix.

6**) Show that every square matrix is uniquely expressible as the sum of a Hermitian matrix and a skew- Hermitian matrix.**

Sol. Let A be any square matrix

Now $\left(A+A^{\theta}\right)^{\theta}=A^{\theta}+\left(A^{\theta}\right)^{\theta}$

$=A^{\theta}+A$

$\left(A+A^{\theta}\right)^{\theta}=A+A^{\theta} \Rightarrow A+A^{\theta}$ is a Hermitian matrix.

$\therefore \frac{1}{2}\left(A+A^{\theta}\right)$ is also a Hermitian matrix

Now $\left(A-A^{\theta}\right)^{\theta}=A^{\theta}-\left(A^{\theta}\right)^{\theta}$

$=A^{\theta}-A=-\left(A-A\right)^{\theta}$

Hence $A-A^{\theta}$ is a skew-Hermitian matrix

$\therefore \frac{1}{2}\left(A-A^{\theta}\right)$ is also a skew –Hermitian matrix.

Uniqueness:

Let A =R+S be another such representation of A

Where R is Hermitian and

S is skew-hermitian

Then $A^{\theta} = (R+S)^{\theta}$

$$= R^{\theta} + S^{\theta}$$

$$= R - S \quad \left(\because R^{\theta} = R, S^{\theta} = -S\right)$$

$$\therefore R = \frac{1}{2}\left(A + A^{\theta}\right) = P \text{ and } S = \frac{1}{2}\left(A - A^{\theta}\right) = Q$$

Hence P=R and Q=S

Thus the representation is unique.

7) Given that A=$\begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, show that $(I-A)(I+A)^{-1}$ is a unitary matrix.

Sol: we have $I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \text{ And}$$

$$I + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$$

$$\therefore (I+A)^{-1} = \frac{1}{1-\left(4i^2-1\right)} \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

Let $B = (I-A)(I+A)^{-1}$

$$B = \frac{1}{6} \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1+(1-2i)(-1-2i) & -1-2i-1-2i \\ 1-2i+1-2i & (-1-2i)(1-2i)+1 \end{bmatrix}$$

$$B = \frac{1}{6} \begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}$$

Now $\overline{B}=\frac{1}{6}\begin{bmatrix}-4 & -2+4i\\ 2+4i & -4\end{bmatrix}$ and $\overline{(B)}^T=\frac{1}{6}\begin{bmatrix}-4 & 2+4i\\ -2+4i & -4\end{bmatrix}$

$$B\left(\overline{B}\right)^T=\frac{1}{36}\begin{bmatrix}-4 & -2-4i\\ 2-4i & -4\end{bmatrix}\begin{bmatrix}-4 & 2+4i\\ -2+4i & -4\end{bmatrix}$$

$$=\frac{1}{36}\begin{bmatrix}36 & 0\\ 0 & 36\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$$

$$\left(\overline{B}\right)^T=B^{-1}$$

i.e., B is unitary matrix.

$\therefore (I-A)(I+A)^{-1}$ is a unitary matrix.

8) Show that the inverse of a unitary matrix is unitary.

Sol: Let A be a unitary matrix. Then $AA^{\theta}=I$

i.e $\left(AA^{\theta}\right)^{-1}=I^{-1}$

$$\Rightarrow\left(A^{\theta}\right)^{-1}A^{-1}=I$$

$$\Rightarrow\left(A^{-1}\right)^{\theta}A^{-1}=I$$

Thus A^{-1} is unitary.

ORTHOGONAL TRANSFORMATION OF A SYMMETRIC MATRIX TODIAGONAL FORM:

Orthogonal matrices:

A square matrix A (with real elements) is said to be orthogonal if $AA^T = A^TA = I$ or $A^T = A^{-1}$

Problems:

1. **Check whether the matrix B is orthogonal. Justify. B =** $\begin{bmatrix}\boldsymbol{cos\,\theta} & \boldsymbol{sin\,\theta} & \mathbf{0}\\ \boldsymbol{-sin\,\theta} & \boldsymbol{cos\,\theta} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{1}\end{bmatrix}$

<u>**Solution**</u>: Condition for orthogonality is $AA^T = A^TA = I$

To prove that: $\boldsymbol{BB^T = B^TB = I}$

B = $\begin{bmatrix}cos\,\theta & sin\,\theta & 0\\ -sin\,\theta & cos\,\theta & 0\\ 0 & 0 & 1\end{bmatrix}$; $B^T = \begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$

$$BB^T = \begin{bmatrix}cos\,\theta & sin\,\theta & 0\\ -sin\,\theta & cos\,\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$
$$= \begin{bmatrix}cos^2\theta + sin^2\theta & -sin\theta cos\theta + sin\theta cos\theta & 0\\ -sin\theta cos\theta + \sin\theta\cos\theta + 0 & sin^2\theta + cos^2\theta + 0 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Similarly,

$$B^TB = \begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}cos\,\theta & sin\,\theta & 0\\ -sin\,\theta & cos\,\theta & 0\\ 0 & 0 & 1\end{bmatrix} =$$
$$\begin{bmatrix}cos^2\theta + sin^2\theta & \sin\theta cos\theta - \sin\theta\cos\theta & 0\\ \sin\theta\cos\theta - \sin\theta\cos\theta & sin^2\theta + cos^2\theta + 0 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Therefore, B is an orthogonal matrix

2. **Show that the matrix P = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogonal**

Solution:To prove that: $PP^T = P^TP = I$

$P = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$; $P^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$PP^T = \begin{bmatrix} cos^2\theta + sin^2\theta & -\sin\theta\cos\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & sin^2\theta + cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \text{I}$$

$$Similarly, P^TP = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$
$$= \begin{bmatrix} cos^2\theta + sin^2\theta & \sin\theta\cos\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \sin\theta\cos\theta & sin^2\theta + cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Therefore, P is an orthogonal matrix

WORKING RULE FOR DIAGONALIZATION

[ORTHOGONAL TRANSFORMATION]:

Step 1: To find the characteristic equation

Step 2: To solve the characteristic equation

Step 3:To find the eigen vectors

Step 4: If the eigen vectors are orthogonal, then form a normalized matrix N

Step 5: Find N^T

Step 6: Calculate AN

Step 7: Calculate D = N^TAN

1. **Diagonalize the matrix $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$**

Solution: Let A = $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = \text{Sum of the main diagonal elements} = 3 + 5 + 3 = 11$

$$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonalelements = (15-1) + (9-1) + (15-1)$$
$$= 14 + 8 + 14 = 36$$

$S_3 = |A| = 3(15-1) + 1(-3+1) + 1(1-5) = 3(14) - 2 - 4 = 42 - 6 = 36$

Therefore, the characteristic equation is $\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

2	1	-11	36	-36
	0	2	-18	36
	1	-9	18	0

$\lambda = 2, \lambda^2 - 9\lambda + 18 = 0 \Rightarrow \lambda = 2, \lambda = \frac{9\pm\sqrt{(-9)^2-4(1)(18)}}{2(1)} = \frac{9\pm\sqrt{81-72}}{2} = \frac{9\pm3}{2} = 6,3$

Hence, the eigen values of A are 2, 3, 6

To find the eigen vectors:

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: When $\lambda = 2$, $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 - x_2 + x_3 = 0$ ---------- (1)

$-x_1 + 3x_2 - x_3 = 0$ -------- (2)

$x_1 - x_2 + x_3 = 0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication,

$x_1 x_2 x_3$

$$\begin{array}{cccc} -1 & 1 & 1 & -1 \\ 3 & -1 & -1 & 3 \end{array}$$

$$\frac{x_1}{1-3} = \frac{x_2}{-1+1} = \frac{x_3}{3-1} \Rightarrow \frac{x_1}{-2} = \frac{x_2}{0} = \frac{x_3}{2} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Case 2: When $\lambda = 3$, $\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$0x_1 - x_2 + x_3 = 0$ ---------- (1)

$-x_1 + 2x_2 - x_3 = 0$ -------- (2)

$x_1 - x_2 + 0x_3 = 0$ --------- (3)

$x_1 x_2 x_3$

$$\begin{array}{cccc} -1 & 1 & 0 & -1 \\ 2 & -1 & -1 & 2 \end{array}$$

$$\frac{x_1}{1-2} = \frac{x_2}{-1-0} = \frac{x_3}{0-1} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Case 3: When $\lambda = 6$, $\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-3x_1 - x_2 + x_3 = 0$ ---------- (1)

$-x_1 - x_2 - x_3 = 0$ -------- (2)

$x_1 - x_2 - 3x_3 = 0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication,

$x_1 x_2 x_3$

$$\begin{array}{cccc} -1 & 1 & -3 & -1 \\ -1 & -1 & -1 & -1 \end{array}$$

$$\frac{x_1}{1+1} = \frac{x_2}{-1-3} = \frac{x_3}{3-1} \Rightarrow \frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{2} \Rightarrow \frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$X_1^T X_2 = [-1 \quad 0 \quad 1]\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = -1 + 0 + 1 = 0$

$X_2^T X_3 = [1 \quad 1 \quad 1]\begin{bmatrix} 1 \\ -2 \end{bmatrix} = 1 - 2 + 1 = 0$

$$X_3^T X_1 = [1 \quad -2 \quad 1]\begin{bmatrix}-1\\0\\1\end{bmatrix} = -1+0+1 = 0$$

Hence, the eigen vectors are orthogonal to each other

The Normalized matrix N = $\begin{bmatrix}\frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\\ \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\end{bmatrix}$; $N^T = \begin{bmatrix}\frac{-1}{\sqrt{2}} & \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}}\end{bmatrix}$

AN = $\begin{bmatrix}3 & -1 & 1\\-1 & 5 & -1\\1 & -1 & 3\end{bmatrix}\begin{bmatrix}\frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\\ \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\end{bmatrix} = \begin{bmatrix}\frac{-2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}}\\ \frac{0}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{-12}{\sqrt{6}}\\ \frac{2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}}\end{bmatrix}$

$$N^T AN = \begin{bmatrix}\frac{-1}{\sqrt{2}} & \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}}\end{bmatrix}\begin{bmatrix}\frac{-2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}}\\ \frac{0}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{-12}{\sqrt{6}}\\ \frac{2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}}\end{bmatrix} = \begin{bmatrix}\frac{4}{2} & \frac{0}{\sqrt{6}} & \frac{0}{\sqrt{12}}\\ \frac{0}{\sqrt{6}} & \frac{9}{3} & \frac{0}{\sqrt{18}}\\ \frac{0}{\sqrt{12}} & \frac{0}{\sqrt{18}} & \frac{36}{6}\end{bmatrix} = \begin{bmatrix}2 & 0 & 0\\0 & 3 & 0\\0 & 0 & 6\end{bmatrix}$$

i.e., $D = N^T AN = \begin{bmatrix}2 & 0 & 0\\0 & 3 & 0\\0 & 0 & 6\end{bmatrix}$

The diagonal elements are the eigen values of A

2. **Diagonalize the matrix $\begin{bmatrix}\mathbf{8} & \mathbf{-6} & \mathbf{2}\\ \mathbf{-6} & \mathbf{7} & \mathbf{-4}\\ \mathbf{2} & \mathbf{-4} & \mathbf{3}\end{bmatrix}$**

Solution: Let A = $\begin{bmatrix}8 & -6 & 2\\-6 & 7 & -4\\2 & -4 & 3\end{bmatrix}$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = \text{Sum of the main diagonal elements} = 8+7+3 = 18$

$$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonalelements = (21-16)+(24-4)+(56-36)$$
$$= 5+20+20 = 45$$

$S_3 = |A| = 8(21-16)+6(-18+8)+2(24-14) = 8(5)-60+20 = 0$

Therefore, the characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda - 0 = 0$ i.e., $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$$\Rightarrow \lambda(\lambda^2 - 18\lambda + 45) = 0 \Rightarrow \lambda = 0, \lambda = \frac{18 \pm \sqrt{(-18)^2 - 4(1)(45)}}{2(1)} = \frac{18 \pm \sqrt{324-180}}{2} = \frac{18 \pm 12}{2}$$
$$= 15, 3$$

Hence, the eigen values of A are 0, 3, 15

To find the eigen vectors:

$(A - \lambda I)X = 0$

$$\begin{bmatrix}8-\lambda & -6 & 2\\-6 & 7-\lambda & -4\\2 & -4 & 3-\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Case 1: When $\boldsymbol{\lambda = 0}$, $\begin{bmatrix}8 & -6 & 2\\-6 & 7 & -4\\2 & -4 & 3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$8x_1 - 6x_2 + 2x_3 = 0$ ----------- (1)

$-6x_1 + 7x_2 - 4x_3 = 0$ --------- (2)

$2x_1 - 4x_2 + 3x_3 = 0$ ---------- (3)

Solving (1) and (2) by rule of cross-multiplication,

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & & \\ -6 & & 2 & & 8 & & -6 & \\ 7 & & -4 & & -6 & & 7 & \end{array}$$

$$\frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36} \Rightarrow \frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20} \Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Case 2: When $\lambda = 3$, $\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$5x_1 - 6x_2 + 2x_3 = 0$ -------- (1)

$-6x_1 + 4x_2 - 4x_3 = 0$ ------- (2)

$2x_1 - 4x_2 + 0x_3 = 0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication,

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & & \\ -6 & & 2 & & 5 & & -6 & \\ 4 & & -4 & & -6 & & 4 & \end{array}$$

$$\frac{x_1}{24-8} = \frac{x_2}{-12+20} = \frac{x_3}{20-36} \Rightarrow \frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16} \Rightarrow \frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$$X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Case 3: When $\lambda = 15$, $\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-7x_1 - 6x_2 + 2x_3 = 0$ ---------- (1)

$-6x_1 - 8x_2 - 4x_3 = 0$ ---------- (2)

$2x_1 - 4x_2 - 12x_3 = 0$ ---------- (3)

Solving (1) and (2) by rule of cross-multiplication,

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & & \\ -6 & & 2 & & -7 & & -6 & \\ -8 & & -4 & & -6 & & -8 & \end{array}$$

$$\frac{x_1}{24+16} = \frac{x_2}{-12-28} = \frac{x_3}{56-36} \Rightarrow \frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20} \Rightarrow \frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$X_1^T X_2 = [1 \quad 2 \quad 2] \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = 2 + 2 - 4 = 0$$

$$X_2^T X_3 = [2 \quad 1 \quad -2] \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = 4-2-2 = 0$$

$$X_3^T X_1 = [2 \quad -2 \quad 1] \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 2 - 4 + 2 = 0$$

Hence, the eigen vectors are orthogonal to each other

The Normalized matrix N = $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

$$N^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

AN =

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} =$$

$$\frac{1}{3}\begin{bmatrix} 8-12+4 & 16-6-4 & 16+12+2 \\ -6+14-8 & -12+7+8 & -12-14-4 \\ 2-8+6 & 4-4-6 & 4+8+3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 & 6 & 30 \\ 0 & 3 & -30 \\ 0 & -6 & 15 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 10 \\ 0 & 1 & -10 \\ 0 & -2 & 5 \end{bmatrix}$$

$$N^T AN = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 & 10 \\ 0 & 1 & -10 \\ 0 & -2 & 5 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0+0+0 & 2+2-4 & 10-20+10 \\ 0+0+0 & 4+1+4 & 20-10-10 \\ 0+0+0 & 4-2-2 & 20+20+5 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 45 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

i.e., $D = N^T AN = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$

The diagonal elements are the eigen values of A

QUADRATIC FORM- REDUCTION OF QUADRATIC FORM TO CANONICAL FORM BY ORTHOGONAL TRANSFORMATION:

Note:

The matrix corresponding to the quadratic form is

$$\begin{bmatrix} coeff.of\ x_1^2 & \frac{1}{2}coeff.of\ x_1x_2 & \frac{1}{2}coeff.of\ x_1x_3 \\ \frac{1}{2}coeff.of\ x_2x_1 & coeff.of\ x_2^2 & \frac{1}{2}coeff.of\ x_2x_3 \\ \frac{1}{2}coeff.of\ x_3x_1 & \frac{1}{2}coeff.of\ x_3x_2 & coeff.of\ x_3^2 \end{bmatrix}$$

1. **Write the matrix of the quadratic form $2x_1^2 - 2x_2^2 + 4x_3^2 + 2x_1x_2 - 6x_1x_3 + 6x_2x_3$**

Solution:Q = $\begin{bmatrix} coeff.of\ x_1^2 & \frac{1}{2}coeff.of\ x_1x_2 & \frac{1}{2}coeff.of\ x_1x_3 \\ \frac{1}{2}coeff.of\ x_2x_1 & coeff.of\ x_2^2 & \frac{1}{2}coeff.of\ x_2x_3 \\ \frac{1}{2}coeff.of\ x_3x_1 & \frac{1}{2}coeff.of\ x_3x_2 & coeff.of\ x_3^2 \end{bmatrix}$

Here $x_2x_1 = x_1x_2$; $x_3x_1 = x_1x_3$; $x_2x_3 = x_3x_2$

$$Q = \begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 3 \\ -3 & 3 & 4 \end{bmatrix}$$

2. **Write the matrix of the quadratic form $2x^2 + 8z^2 + 4xy + 10xz - 2yz$**

Solution: Q = $\begin{bmatrix} coeff.of\ x^2 & \frac{1}{2}\ coeff.of\ xy & \frac{1}{2}\ coeff.of\ xz \\ \frac{1}{2}\ coeff.of\ yx & coeff.of\ y^2 & \frac{1}{2}\ coeff.of\ yz \\ \frac{1}{2}\ coeff.of\ zx & \frac{1}{2}\ coeff.of\ zy & coeff.of\ z^2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 5 \\ 2 & 0 & -1 \\ 5 & -1 & 8 \end{bmatrix}$

3. **Write down the quadratic form corresponding to the following symmetric matrix** $\begin{bmatrix} \mathbf{0} & \mathbf{-1} & \mathbf{2} \\ \mathbf{-1} & \mathbf{1} & \mathbf{4} \\ \mathbf{2} & \mathbf{4} & \mathbf{3} \end{bmatrix}$

Solution:Let $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ -1 & 1 & 4 \\ 2 & 4 & 3 \end{bmatrix}$

The required quadratic form is

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2(a_{12})x_1x_2 + 2(a_{23})x_2x_3 + 2(a_{13})x_1x_3$$

$= 0x_1^2 + x_2^2 + 3x_3^2 - 2x_1x_2 + 4x_1x_3 + 8x_2x_3$

1. **Determine the nature of the following quadratic form f$(x_1, x_2, x_3) = x_1^2 + 2x_2^2$**

Solution:The matrix of the quadratic form is Q = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The eigen values of the matrix are 1, 2, 0

Therefore, the quadratic form is Positive Semi-definite

2. **Discuss the nature of the quadratic form $2x^2 + 3y^2 + 2z^2 + 2xy$ without reducing it to canonical form**

Solution:The matrix of the quadratic form is Q = $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$D_1 = 2(+ve)$

$D_2 = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5(+ve)$

$D_3 = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 2(6-0) - 1(2-0) + 0 = 12 - 2 = 10(+ve)$

Therefore, the quadratic form is positive definite

REDUCTION OF QUADRATIC FORM TO CANONICAL FORM THROUGH ORTHOGONAL TRANSFORMATION [OR SUM OF SQUARES FORM OR PRINCIPAL AXES FORM]

Working rule:

Step 1: Write the matrix of the given quadratic form

Step 2: To find the characteristic equation

Step 3: To solve the characteristic equation

Step 4: To find the eigen vectors orthogonal to each other

Step 5: Form the Normalized matrix N

Step 6: Find N^T

Step 7: Find AN

Step 8: Find D = $N^T AN$

Step 9: The canonical form is $[y_1 y_2 y_3][D]\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

PROBLEMS:

1. **Reduce the given quadratic form Q to its canonical form using orthogonal transformation Q = $x^2 + 3y^2 + 3z^2 - 2yz$**

Solution: The matrix of the Q.F is A = $\begin{bmatrix} coeff.of\ x^2 & \frac{1}{2}\ coeff.of\ xy & \frac{1}{2}\ coeff.of\ xz \\ \frac{1}{2}\ coeff.of\ yx & coeff.of\ y^2 & \frac{1}{2}\ coeff.of\ yz \\ \frac{1}{2}\ coeff.of\ zx & \frac{1}{2}\ coeff.of\ zy & coeff.of\ z^2 \end{bmatrix}$

i.e., A = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 3 + 3 = 7$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elemeents = (9-1) + (3-0) + (3-0)$
$= 8 + 3 + 3 = 14$

$S_3 = |A| = 1(9-1) + 0 + 0 = 8$

The characteristic equation of A is $\lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$

1	1	-7	14	-8
	0	1	-6	8
	1	-6	8	0

$$\lambda = 1, \lambda^2 - 6\lambda + 8 = 0 \Rightarrow \lambda = 1, \lambda = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(8)}}{2(1)} = \frac{6 \pm \sqrt{4}}{2} = \frac{6 \pm 2}{2} = 4, 2$$

The eigen values are 1, 2, 4

To find the eigen vectors:

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: When $\lambda = 1$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$0x_1 + 0x_2 + 0x_3 = 0$ --------- (1)

$0x_1 + 2x_2 - x_3 = 0$ --------- (2)

$0x_1 - x_2 + 2x_3 = 0$ --------- (3)

Solving (2) and (3) by rule of cross multiplication, we get,

$x_1 x_2 x_3$

$$\begin{array}{cccc} 2 & -1 & 0 & 2 \\ -1 & 2 & 0 & -1 \end{array}$$

$$\frac{x_1}{4-1} = \frac{x_2}{0-0} = \frac{x_3}{0-0} \Rightarrow \frac{x_1}{3} = \frac{x_2}{0} = \frac{x_3}{0} \Rightarrow \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0}$$

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Case 2: When $\lambda = 2$, $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-x_1 + 0x_2 + 0x_3 = 0$ -------- (1)

$0x_1 + x_2 - x_3 = 0$ -------- (2)

$0x_1 - x_2 + x_3 = 0$ -------- (3)

Solving (1) and (2) by rule of cross multiplication, we get,

$x_1 x_2 x_3$

0 0 -1 0

1 -1 0 1

$$\frac{x_1}{0-0} = \frac{x_2}{0-1} = \frac{x_3}{-1-0} \Rightarrow \frac{x_1}{0} = \frac{x_2}{-1} = \frac{x_3}{-1} \Rightarrow \frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Case 3: When $\lambda = 4$, $\begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-3x_1 + 0x_2 + 0x_3 = 0$ -------- (1)

$0x_1 - x_2 - x_3 = 0$ -------- (2)

$0x_1 - x_2 - x_3 = 0$ -------- (3)

Solving (1) and (2) by rule of cross multiplication, we get,

$x_1 x_2 x_3$

0 0 -3 0

-1 -1 0 -1

$$\frac{x_1}{0-0} = \frac{x_2}{0-3} = \frac{x_3}{3-0} \Rightarrow \frac{x_1}{0} = \frac{x_2}{-3} = \frac{x_3}{3} \Rightarrow \frac{x_1}{0} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$X_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

The Normalized matrix N = $\begin{bmatrix} \frac{1}{\sqrt{1}} & \frac{0}{\sqrt{2}} & \frac{0}{\sqrt{2}} \\ \frac{0}{\sqrt{1}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{0}{\sqrt{1}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}; N^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

AN = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{1}} & \frac{0}{\sqrt{2}} & \frac{0}{\sqrt{2}} \\ \frac{0}{\sqrt{1}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{0}{\sqrt{1}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{2}} & \frac{-4}{\sqrt{2}} \\ 0 & \frac{2}{\sqrt{2}} & \frac{4}{\sqrt{2}} \end{bmatrix}$

$$N^T AN = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{2}} & \frac{-4}{\sqrt{2}} \\ 0 & \frac{2}{\sqrt{2}} & \frac{4}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

i.e., $D = N^T AN = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

The canonical form is $[y_1 y_2 y_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y_1^2 + 2y_2^2 + 4y_3^2$

canonical form is $[y_1 \quad y_2 \quad y_3] \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 8y_1^2 + 2y_2^2 + 2y_3^2$

1. **Reduce the quadratic form to a canonical form by an orthogonal reduction $2x_1x_2 + 2x_1x_3 - 2x_2x_3$. Also find its nature.**

Solution: The matrix of the Q.F is A = $\begin{bmatrix} coeff.of\ x_1^2 & \frac{1}{2}coeff.of\ x_1x_2 & \frac{1}{2}coeff.of\ x_1x_3 \\ \frac{1}{2}coeff.of\ x_2x_1 & coeff.of\ x_2^2 & \frac{1}{2}coeff.of\ x_2x_3 \\ \frac{1}{2}coeff.of\ x_3x_1 & \frac{1}{2}coeff.of\ x_3x_2 & coeff.of\ x_3^2 \end{bmatrix}$

i.e., A = $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 0$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements = -1 - 1 - 1 = -3$

$S_3 = |A| = 0(0-1) - 1(0+1) + 1(-1-0) = 0 - 1 - 1 = -2$

The characteristic equation of A is $\lambda^3 - 0\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda^3 - 3\lambda + 2 = 0$

$$\begin{array}{c|cccc} 1 & 1 & 0 & -3 & 2 \\ & 0 & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & 0 \end{array}$$

$$\lambda = 1, \lambda^2 + \lambda - 2 = 0 \Rightarrow \lambda = 1, \lambda = \frac{-1 \pm \sqrt{1^2 - 4(1)(-2)}}{2(1)} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} = -2, 1$$

The eigen values are 1, 1, -2

To find the eigen vectors:

$(A - \lambda I)X = 0$

$$\begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & -1 \\ 1 & -1 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: When $\lambda = -2$, $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$2x_1 + x_2 + x_3 = 0$ ------- (1)

$x_1 + 2x_2 - x_3 = 0$ ------- (2)

$x_1 - x_2 + 2x_3 = 0$ ------- (3)

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & & 1 & & 2 & & 1 \\ 2 & & -1 & & 1 & & 2 \end{array}$$

$$\frac{x_1}{-1-2} = \frac{x_2}{1+2} = \frac{x_3}{4-1} \Rightarrow \frac{x_1}{-3} = \frac{x_2}{3} = \frac{x_3}{3} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Case 2: When $\lambda = 1$, $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-x_1 + x_2 + x_3 = 0$ ------- (1)

$x_1 - x_2 - x_3 = 0$ ------ (2)

$x_1 - x_2 - x_3 = 0$ ------ (3)

All three equations are one and the same.

Put $x_1 = 0$, $x_2 = -x_3$. Let $x_3 = 1$. Then $x_2 = -1$

$$X_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Let $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$. Since X_3 is orthogonal to X_1 *and* X_2, $X_1^T X_3 = 0$ *and* $X_2^T X_3 = 0$

$$[-1 \quad 1 \quad 1]\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \; and \; [0 \quad -1 \quad 1]\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

$-l + m + n = 0$ -------- (1)

$0l - m + n = 0$ --------- (2)

$$\begin{array}{cccccccc} & l & & m & & n & \\ 1 & & 1 & & -1 & & 1 \\ -1 & & 1 & & 0 & & -1 \end{array}$$

$$\frac{l}{1+1} = \frac{m}{0+1} = \frac{n}{1-0} \Rightarrow \frac{l}{2} = \frac{m}{1} = \frac{n}{1}$$

$$X_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

The Normalized matrix N = $\begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{0}{\sqrt{2}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$; $N^T = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{0}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

AN = $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{0}{\sqrt{2}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{3}} & \frac{0}{\sqrt{2}} & \frac{2}{\sqrt{6}} \\ \frac{-2}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

$$N^T AN = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{0}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}\begin{bmatrix} \frac{2}{\sqrt{3}} & \frac{0}{\sqrt{2}} & \frac{2}{\sqrt{6}} \\ \frac{-2}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{-6}{3} & \frac{0}{\sqrt{6}} & \frac{0}{\sqrt{18}} \\ \frac{0}{\sqrt{6}} & \frac{2}{2} & \frac{0}{\sqrt{12}} \\ \frac{0}{\sqrt{18}} & \frac{0}{\sqrt{12}} & \frac{6}{6} \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i.e., $D = N^T AN = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The canonical form is $[y_1 \quad y_2 \quad y_3]\begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = -2y_1^2 + y_2^2 + y_3^2$

Nature: The eigen values are -2, 1, 1. Therefore, it is indefinite in nature.

Important Questions

1. Define the rank of the matrix **(JNTUA JUNE 2012,13,14,15)**

2. Echelon form and Rank of the matrix

i) Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ into echelon form and hence find its rank

ii) Reduce the matrix Reduce the matrix $\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$ into Echelon form and hence find its rank

iii) Reduce the matrix $A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$ into echelon form and hence find its rank **(JNTUA JUNE 2011)**

iv) Define the rank of the matrix and find the rank of the matrix $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$ **(JNTUA MAY 2005,06. SEP 2008)**

v) Determine the rank of the matrix $A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

vi) Find the value of k such that the rank of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & k & 7 \\ 3 & 6 & 10 \end{bmatrix}$ is 2 **(JNTUA 2006)**

vii) Find the rank of $\begin{bmatrix} 2 & -4 & 3 & -1 & 0 \\ 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$ **(JNTUA 2008,09)**

viii) Find the rank of the matrix $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ **(JNTUA 2008)**

3 .NORMAL FORM (OR) CANONICAL FORM

i) Reduce the matrix to canonical form and find its rank $A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 8 & 3 & 7 & 5 \\ 8 & 5 & 11 & 6 \end{bmatrix}$

ii) Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$ to canonical form (normal form) and find its rank **(JNTUA JUNE 2009)**

iii) Reduce the matrix $A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ to normal form and find its rank **(JNTUA 2008)**

v) Find the rank of the matrix $A = \begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ by reducing it to it to canonical form **(JNTUA 2006,SEP 2008)**

v) By reducing the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & 10 \end{bmatrix}$ into normal form and find its rank **(JNTUA 2002)**

vi) Reduce A to normal form and find its rank $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ **(JNTUA 2002)**

vii) Reduce the matrix to canonical form and find its rank $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ **(JNTUA 2005)**

4.Find two non-singular matrices P and Q such that PAQ is in the normal form form of the following matrices and find the rank of the following matrices.

i) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$ *ii)* $A = \begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}$ iii) $A = \begin{bmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{bmatrix}$

iv) $A = \begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}$ **(JNTUA 2005)**

v) $A = \begin{bmatrix} 2 & 1 & -3 & 6 \\ 2 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ vi) $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 0 \end{bmatrix}$ **(JNTUA MAY 2005)**

5.NON- HOMOGENOUS SYSTEM OF LINEAR EQUATIONS (AX =B)

i) Find the following system of equations are consistent if so solve them
$x + 2y + 2z = 2,\ 3x - 2y - z = 5,\ 2x - 5y + 3z = -4,\ x + 4y + 6z = 0$ **(JNTUA 2001,02,04,05)**

ii) Find whether the following equations are consistent, if solve them .
$x + y + 2z = 4;\ 2x - y + 3z = 9; 3x - y - z = 2$ **(JNTUA MAY 2005)**

iii) Find the value of λ for which the system of equations
$3x - y + 4z = 3;\ x + 2y - 3z = -2;\ x + 2y - 3z = -2$ Will have infinite number of solutions and solve them with that λ value

iv) Find whether the following set of equations are consistent if so, solve them.$x_1 + x_2 + x_3 + x_4 = 0$,

$x_1 + x_2 + x_3 - x_4 = 4, .x_1 + x_2 - x_3 + x_4 = -4, .x_1 - x_2 + x_3 + x_4 = 2$ **(JNTUA MAY 2005)**

v) Prove that the following set of equations are consistent and solve them.
$3x + 3y + 2z = 1; x + 2y = 4; 10y + 3z = -2; 2x - 3y - z = 5$ **(JNTUA MAY 2006,07,08(K),09(H),09(K),10)**

vi) Solve $x + y + z = 6; x - y + 2z = 5; 2x - 2y + 3z = 7$ **(JNTUA 2008)**

vii) Test for consistency and solve $2x + 3y + 7z = 5; 3x + y - 3z = 12; 2x + 19y - 47z = 32$

viii) Find the values of a and b for which the equations
$x + ay + z = 3; x + 2y + 2z = b; x + 5y + 3z = 9$ are consistent. when will these equations have a unique solution? **(JNTUA 2004)**

6.HOMOGENOUS SYSTEM OF LINEAR EQUATIONS (AX =O)

i)Solve the system of equations $x + 3y - 2z = 0; 2x - y + 4z = 0; x - 11y + 14z = 0$

ii)Solve the system of equations $x + y - 3z + 2w = 0;\ \ 2x - y + 2z - 3w = 0;$

$3x - 2y + z - w = 0;\ -4x + y - 3z + w = 0$

iii)Show that the only real number λ $f$or which for which the system $x + 2y + 3z = \lambda x;\ 3x + y + 2z = \lambda y;\ 2x + 3y + z = \lambda z$ has non-zero solution is 6 and solve them, when $\lambda = 6$ **(JNTUA 2005,06,08)**

iv)Determine whether the following equations will have a non-trivial solution if so solve them. $4x + 2y + z + 3w = 0;\ \ 6x + 3y + 4z + 7w = 0;\ 2x + y + w = 0$ **(JNTUA MAY 2006)**

v) Solve the system of equations $x + y + w = 0;\ y + z = 0;\ x + y + z + w = 0;\ x + y + 2z = 0$ **(JNTUA 2008,09)**

vi) Solve the system of equations $2x - y + 3z = 0;\ 3x + 2y + z = 0\ and\ x - 4y + 5z = 0$ **(JNTUA 2008)**

vii)Find all the solutions of system of equations $x + 2y - z = 0;\ 2x + y + z = 0\ and\ x - 4y + 5z = 0$ **(JNTUA 2008)**

viii)Solve completely the system of equations $x + 3y - 2z = 0;\ 2x - y + 4z = 0\ and\ x - 11y + 14z = 0$

EIGEN VALUES AND EIGEN VECTORS (JNTUA 2009)

1.Find the characteristic roots of the matrix $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ **(JNTUA 2008,09)**

II. Find the eigen values and the corresponding eigen vectors of the following matrices

i) $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ **(JNTUA MAY 2006,08,12)** ii) $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

iii) $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ **(JNTUA MAY 2006,08)** iv) $A = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$ **(JNTUA 2001,06)**

v) $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ **(JNTUA 2005)** vi) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ **(JNTUA 2006)**

vii) $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ **(JNTUA 2008)** vii) $A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$ **(JNTUA 2008,10)**

ix)Verify that the sum of eigen values is equal to trace of $'A'$ for the matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ **(JNTUA 2007)** and find the corresponding eigen vectors x) $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ **(JNTUA 2009)**

III. DIAGONALIZATION AND CALUCULATION OF POWERS OF A MATRIX

i) Determine the modal matrix $P\ for\ A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ and hence diagonalize A .

ii) Diagonalize the matrix $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ and find A^4 **(JNTUA 2006)**

iii) If A= $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$ find a) A^8 b) A^4 **(JNTUA 2006)**

iv) Diagonalize the matrix A = $\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ **(JNTUA 2004,09)**

v) Find a matrix P which transforms the matrix, $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ to diagonal form J(A) Dec2016

vi) Diagonalize the matrix A = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ and find A^5

vii) If A=$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, $find\ A^{50}$

IV.CAYLEY-HAMILTON THEOREM AND ITS PROBLEMS

i) State and prove Cayley- Hamilton theorem

ii) Verify Cayley-Hamilton theorem for the matrix A= $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ .Hence find A^{-1} **(JNTUA 2005,06)**

iii) Find the inverse of the matrix A = $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ by using Cayley-Hamilton theorem iv)State Cayley-Hamilton theorem and use it to find the inverse of the matrix A = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$ **(JNTUA 2001)**

v)Using Cayley-Hamilton theorem find the inverse and A^4 of the matrix A = $\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$ **(JNTUA 2002)**

vi)Show that the matrix A = $\begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$ satisfies its characteristic equation . Hence find A^{-1}**(JNTUA 2002)**

vii) Using Cayley-Hamilton theorem find the inverse and A^8 of the matrix A = $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ **(JNTUA 2003)**

viii) If A = $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ Verify Cayley-Hamilton theorem **(JNTUA 2006)**

ix) Verify Cayley-Hamilton theorem and find the characteristic roots where A = $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ **(JNTUA 2009)**

x) Verify Cayley-Hamilton theorem for the matrix A= $\begin{bmatrix} 3 & 4 & 1 \\ 2 & 1 & 6 \\ -1 & 4 & 7 \end{bmatrix}$ .Hence find A^{-1} **(JNTUA 2009)**

COMPLE X MATRICES

i)Define Hermitian and Skew-Hermitian and Unitary matrices**(JNTUA 2002,04,10,11,15)**

I.Properties of Hermitian and Skew-Hermitian and Unitary matrices

i)The eigen values of a Hermitian matrix are all real **(JNTUA 2002,04,10,11,12,15)**

ii) The eigen values of a Skew-Hermitian matrix are purely imaginary or zero**(JNTUA 2002)**

iii) The eigen values of Unitary matrix have absolute value $'1'$**(JNTUA 2002,03)**

iv)The inverse and transpose of an unitary matrix is unitary **(JNTUA 2002)**

v)If $'A'$ is any square matrix then prove that

a)$A + A^{\theta}$ is Hermitian matrix (b) $A.A^{\theta}$, $A^{\theta}.A$ are Hermitian matrices (c) $A - A^{\theta}$ is a Skew-Hermitian matrix **(JNTUA 2009)**

vi)Every square matrix A can be expressed as $P + i\,Q\ where\ \ P, Q\ \ are$ Hermitian matrices

II.Problems of Hermitian and Skew-Hermitian and Unitary matrices(JNTUA 2005)

Find the eigen values of the following matrices

i) $A=\begin{pmatrix}4 & 1-3i\\ 1+3i & 7\end{pmatrix}$ **(JNTUA 2006)** ii) $A=\begin{pmatrix}3i & 2+i\\ -2+i & -i\end{pmatrix}$ iii) $A=\begin{bmatrix}\frac{i}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{i}{2}\end{bmatrix}$

iv) Find the eigen values and eigen vectors of the Hermitian matrix $\begin{pmatrix}2 & 3+4i\\ 3-4i & 2\end{pmatrix}$**(JNTUA 2006)**

v)S.T A = $\begin{bmatrix}i & 0 & 0\\ 0 & 0 & i\\ 0 & i & 0\end{bmatrix}$ is a Skew- Hermitian matrix and also Unitary. Find the eigen values and eigen vectors of A**(JNTUA 2006)**

vi) Show that A = $\begin{pmatrix}a+ic & -b+di\\ b+di & a-ic\end{pmatrix}$ is unitary if $a^2+b^2+c^2+d^2=1$**(JNTUA 2004)**

QUADRATIC FORMS

i)Define Quadratic form

ii)Define Index, Signature, Nature of Quadratic form

I.Find the symmetric matrix of the following Quadratic forms

i)$x_1^2+6x_1x_2+5x_2^2$ ii) $x^2+2y^2+3z^2+4xy+5yz+6zx$

iii)$x_1^2+2x_2^2-7x_3^2-4x_1x_2+8x_1x_3+5x_2x_3$ **(JNTUA 2003)** iii)$x_1^2+2x_2^2-7x_3^2-4x_1x_2+8x_1x_3+5x_2x_3$

iv) $x_1^2+2x_2^2+4x_2x_3+x_3x_4$

II.Find the Quadratic form relating to to the following matrices

i) A = $\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}$ ii) A = $\begin{bmatrix}1 & 2 & 3\\ 2 & 1 & 3\\ 3 & 3 & 1\end{bmatrix}$ iii) A = $\begin{bmatrix}1 & 2 & 5\\ 2 & 0 & 3\\ 5 & 3 & 4\end{bmatrix}$ iv) A = $\begin{bmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{bmatrix}$ **(JNTUA 2003)**

III.Reduce the following Quadratic forms to Canonical form (or) normal form (or) sum of squares form by using Linear Transformation $X=PY$ and also find Rank, Index, Signature, Nature

i)$10x^2+2y^2+5z^2-4xy+6yz-10xz$ **(JNTUA 2007,08)** (ii) $4x^2+3y^2+z^2-8xy-6yz+4xz$**(JNTUK 2008)**

iii)$2x_1^2+x_2^2-3x_3^2+12x_1x_2-4x_1x_3-8x_2x_3$ iv) $7x^2+6y^2+5z^2-4xy-4yz$

v) $x^2+4y^2+9z^2+t^2-4xy-12yz+6xz-2xt-6zt$ vi) $2x^2+2y^2+2z^2-2xy-2yz-2xz$**(JNTUA 2008)**

vii) $3x^2-2y^2-z^2-4xy+12yz+8zx$**(JNTUH 2009)**

IV. .Reduce the following Quadratic forms to Canonical form (or) normal form (or) sum of squares form by using Orthogonal Transformation $X=PY$ *where P is an orthogonal matrix* and give the matrix of transformation. And also find Rank, Index, Signature, Nature.

i) $3x^2+2y^2+3z^2-2xy-2yz$ ii) $3x_1^2+3x_2^2+3x_3^2+2x_1x_2+2x_1x_3-2x_2x_3$ iii) $x_1^2+3x_2^2+3x_3^2-2x_2x_3$ iv) $2x^2+2y^2+2z^2-2xy-2yz-2xz$

v) $5x^2+26y^2+10z^2+6xy+4yz+14zx$ vi) $3x_1^2+5x_2^2+3x_3^2-2x_1x_2+2x_1x_3-2x_2x_3$

vii) $3x^2+5y^2+3z^2-2xy-2yz+2xz$ **(JNTUA 2005,06,08,10)** viii) $6x^2+3y^2+3z^2-4xy-2yz+4xz$**(JNTUH 2011, JNTUA 2009)**

V.Reduce the following Quadratic forms to Canonical form (or) normal form (or) sum of squares form by using Lagrange's reduction and also find Rank, Index, Signature, Nature

i) $x_1^2+2x_2^2-7x_3^2-4x_1x_2+2x_1x_3$ ii) $x^2+y^2+2z^2-2xy+4yz+4xz$**(JNTUA 2009)**

iii) $2x^2+2y^2+2z^2-2xy-2yz-2xz$ iv) $2x_1^2+7x_2^2+5x_3^2-8x_1x_2-10x_2x_3+4x_1x_3$

v) $6x_1^2+3x_2^2+3x_3^2-4x_1x_2-2x_2x_3+4x_1x_3$**(JNTUK 2011)**

VI.Identify the nature of the following Quadratic forms

$i)\ {x_1}^2 + 3{x_2}^2 + 3{x_3}^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$ ii) $3x^2 + 2y^2 + 3z^2 - 2xy - 2yz$

iii) $2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2xz$ iv) $2{x_1}^2 + 7{x_2}^2 + 5{x_3}^2 - 8x_1x_2 - 10x_2x_3 + 4x_1x_3$

v)identify the nature of the matrix A = $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ **(JNTUA 2000,2009)**

What is the rank of the matrix $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 1 & -2 & 3 & 0 \\ 0 & 0 & 4 & 8 \\ 2 & 4 & 0 & 6 \end{bmatrix}$ J(A) Dec 2016

2.Explain unitary matrix with proper example J(A) Dec 2016

3.Find the characteristic equation of the matrix A = $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ and hence find its inverse use cayley-Hamilton theorem J(A) Dec 2016

4. Find the Eigen vectors of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ and hence reduce the quadratic form ${x_1}^2 + 3{x_2}^2 + 3{x_3}^2 - 2x_2x_3$ to a sum of squares also write its nature J(A) 2015

5.Determine the value of λ for which the following system of equations has non trivial solutions and find them

$(\lambda-1)x+(3\lambda+1)y+2\lambda z=0$, $(\lambda-1)x+(4\lambda-2)y+(\lambda+3)z=0$, $2x+(3\lambda+1)y+3(\lambda-1)z=0$ J(A) 2015