8	-0.0422	-0.0539	-0.0426	-0.0539	-0.0695	-0.0543	-0.0426	-0.0543	-0.0428
9	-0.0426	-0.0543	-0.0428	-0.0543	-0.0699	-0.0545	-0.0428	-0.0545	-0.0429
10	-0.0428	-0.0545	-0.0429	-0.0545	-0.0701	-0.0546	-0.0429	-0.0546	-0.0429
11	-0.0429	-0.0546	-0.0429	-0.0546	-0.0702	-0.0546	-0.0429	-0.0546	-0.0429
12	-0.0429	-0.0546	-0.0429	-0.0546	-0.0703	-0.0547	-0.0429	-0.0547	-0.0430
13	-0.0429	-0.0547	-0.0430	-0.0547	-0.0703	-0.0547	-0.0430	-0.0547	-0.0430
14	-0.0430	-0.0547	-0.0430	-0.0547	-0.0703	-0.0547	-0.0430	-0.0547	-0.0430
15	-0.0430	-0.0547	-0.0430	-0.0547	-0.0703	-0.0547	-0.0430	-0.0547	-0.0430

UNIT IV

Curve Fitting and Method of Least Squares

Curve Fitting

Curve fitting is the process of introducing mathematical relationships between dependent and independent variables in the form of an equation for a given set of data.

Method of Least Squares

The method of least squares helps us to find the values of unknowns aa and bb in such a way that the following two conditions are satisfied:

- The sum of the residual (deviations) of observed values of Y and corresponding expected (estimated) values of YY will be zero. $\sum(Y-\hat{Y})$
- The sum of the squares of the residual (deviations) of observed values of YY and corresponding expected values $(\hat{Y})$ should be at least $\sum(Y-\hat{Y})^2$

Fitting of a Straight Line

A straight line can be fitted to the given data by the method of least squares. The equation of a straight line or least square line is Y=a+bXY

=a+bX,

where a and b are constants or unknowns.

To compute the values of these constants we need as many equations as the number of constants in the equation. These equations are called normal equations. In a straight line there are two constants aa and bb so we require two normal equations.

Normal Equation for '*a*' $\sum Y=na+b\sum X$

Normal Equation for '*b*' $\sum XY=a\sum X+b\sum X^2$

The direct formula of finding a and b is written as

Example:

Fit a least square line for the following data. Also find the trend values and show that $\sum(Y-\hat{Y})=0$

X	1	2	3	4	5
Y	**2**	**5**	**3**	**8**	**7**

Solution:

X	Y	XY	X^2	$\hat{Y}=1.1+1.3X$	Y−Y
1	**2**	**2**	**1**	**2.4**	**-0.4**
2	**5**	**10**	**4**	**3.7**	**+1.3**
3	**3**	**9**	**9**	**5.0**	**-2**
4	**8**	**32**	**16**	**6.3**	**1.7**
5	**7**	**35**	**25**	**7.6**	**-0.6**
$\sum X=15$	$\sum Y=25$	$\sum XY=88$	$\sum X^2=55$	**Trend Values**	$\sum(Y-\hat{Y})=0$

The equation of least square line $Y=a+bX$

Normal equation for '*a*' $\sum Y=na+b\sum X$

$$25=5a+15b\sum Y$$

$$25=5a+15b \text{ ---- } (1)$$

Normal equation for'*b*' $\sum XY=a\sum X+b\sum X^2$

$$88=15a+55b\sum XY$$

$$88=15a+55b \text{ ----} (2)$$

Eliminate a from equation (1) and (2), multiply equation (2) by **3** and subtract from equation (2). Thus we get the values of a and b.

Here $a=1.1$. and $b=1.3$ the equation of least square line becomes $Y=1.1+1.3X$.

Numerical Example Fitting of a second degree parabola

Fit a parabola of second degree to th following data

X	0	1	2	3	4
Y	1	1.8	1.3	2.5	6.3

Sol;

X	Y	X^2	X^3	X^4	XY	X^2Y
0	1	0	0	0	0	0
1	1.8	1	1	1	1	1
2	1.3	4	8	16	2.6	5.2
3	2.5	9	27	81	7.5	22.5
4	6.3	16	64	256	25.2	100.8
ΣX=10	ΣY=12.9	$\Sigma X^2=30$	$\Sigma X^3=100$	$\Sigma X^4=354$	ΣXY=37.1	$\Sigma X^2Y=130.3$

$$\sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i + c\sum_{i=1}^{n} x_i^{2}$$

$$12.9 = 5a + 10b + 30c$$

$$\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^{2} + c\sum_{i=1}^{n} x_i^{3}$$

$$37.1 = 10a + 30b + 100c$$

$$\sum_{i=1}^{n} x_i^{2} y_i = a\sum_{i=1}^{n} x_i^{2} + b\sum_{i=1}^{n} x_i^{3} + c\sum_{i=1}^{n} x_i^{4}$$

$$130.3 = 30a + 100b + 354c$$

Numerical Example Fitting of a second degree parabola

$$a=\frac{\begin{vmatrix}\sum y_i & \sum x_i & \sum x_i^2\\ \sum x_i y_i & \sum x_i^2 & \sum x_i^3\\ \sum x_i^2 y_i & \sum x_i^3 & \sum x_i^4\end{vmatrix}}{\begin{vmatrix} n & \sum x_i & \sum x_i^2\\ \sum x_i & \sum x_i^2 & \sum x_i^3\\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4\end{vmatrix}}=\frac{\begin{vmatrix}12.9 & 10 & 30\\ 37.1 & 30 & 100\\ 130.3 & 100 & 354\end{vmatrix}}{\begin{vmatrix}5 & 10 & 30\\ 10 & 30 & 100\\ 30 & 100 & 354\end{vmatrix}}=\frac{994}{700}=1.42$$

$$\mathrm{b}=\frac{\begin{vmatrix} n & \sum y_i & \sum x_i^2\\ \sum x_{ii} & \sum x_i y_i & \sum x_i^3\\ \sum x_i^2 & \sum x_i^2 y_i & \sum x_i^4\end{vmatrix}}{\begin{vmatrix} n & \sum x_i & \sum x_i^2\\ \sum x_i & \sum x_i^2 & \sum x_i^3\\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4\end{vmatrix}}=\frac{\begin{vmatrix}5 & 12.9 & 30\\ 10 & 37.1 & 100\\ 30 & 130.3 & 354\end{vmatrix}}{\begin{vmatrix}5 & 10 & 30\\ 10 & 30 & 100\\ 30 & 100 & 354\end{vmatrix}}=\frac{-749}{700}=-1.07$$

Subsituting for ‘a’ and ‘b’in this equation we get ‘c

$$\sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i + c\sum_{i=1}^{n} x_i^2$$

$$12.9 = 5a + 10b + 30c$$

$$12.9 = 5(1.42) + 10(-1.07) + 30c$$

$$c = 0.55$$

$$\therefore\ y = 1.42 - 1.07x + 0.55\,x^2$$

Fitting of an exponential curve

The equation to be fitted is $y = ab^x$

Steps:

$$\log_{10} y = \log_{10} a + x \log_{10} b$$
$$U = A + XB$$
$$\sum U = nA + B \sum X$$
$$\sum UX = A \sum X + B \sum X^2$$

Solve and Find A & B

then,

a = antilog(A)

b = antilog(B)

Numerical Example-
Fitting of a exponential curve

X	Y
1	1
2	1.2
3	1.8
4	2.5
5	3.6
6	4.7
7	6.6
8	9.1

X	Y	U=logY	UX	X^2
1	1	0	0	1
2	1.2	0.301	0.3612	4
3	1.8	0.4771	0.8588	9
4	2.5	0.602	1.5051	16
5	3.6	0.6989	2.5163	25
6	4.7	0.7782	3.6573	36
7	6.6	0.8451	5.5776	49
8	9.1	0.9031	8.2181	64
ΣX=36		ΣU=3.73	ΣUX=22.694	ΣX^2=204

Substituting the values

$$\sum U = nA + B\sum X$$
$$\sum UX = A\sum X + B\sum X^2$$

we get, A = -0.168, B= 0.141

$$Y = 0.68 * 1.38^{X}$$

Numerical integration using Simpson's 1/3 and

Simpson's 3/8 rules

1.*Calculus* is the mathematics of change. Since engineers continuously deal with systems and processes that change, *calculus* is an essential tool of engineering.

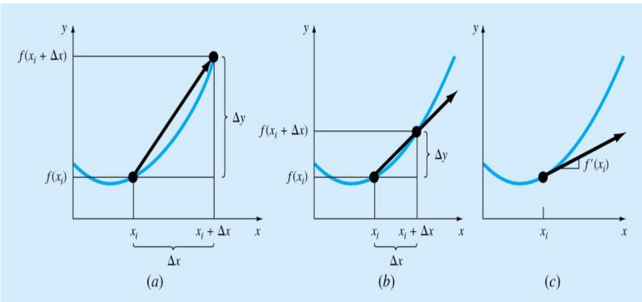
2.Standing at the heart of *calculus* are the concepts of:

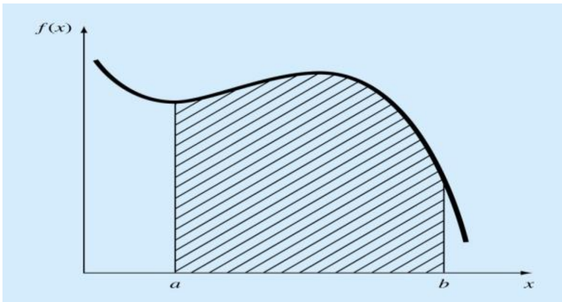
Differentiation

$$\frac{\Delta y}{\Delta x} = \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}$$
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}$$

Integration

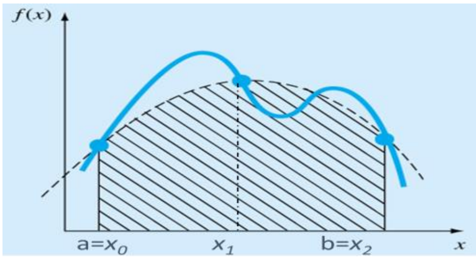
$$I = \int_a^b f(x)dx$$





Simpson's 1/3 Rule

More accurate estimate of an integral is obtained if a high-order polynomial



is used to connect the points.

These formulas are called *Simpson's rules*.

Simpson's 1/3 Rule: results when a *2nd order Lagrange interpolating polynomial* is used for *f(x)*

$$I = \int_a^b f(x)dx \cong \int_a^b f_2(x)dx \qquad \text{where } f_2(x) \text{ is a second-order polynomial.}$$

Using $a = x_0 \quad b = x_2$

$$I = \int_{x_0}^{x_2} \left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \right] dx$$

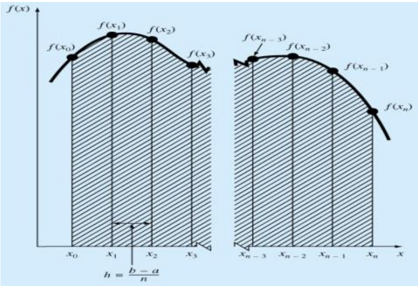
after integration and algebraic manipulation, the following formula results :

$$I \cong \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2) \right] \qquad h = \frac{b-a}{2} \qquad \Leftarrow \quad \text{SIMPSON'S 1/3 RULE}$$

The Multiple-Application Simpson's 1/3 Rule

Just as the trapezoidal rule, Simpson's rule can be improved by dividing the integration interval into a number of segments of equal width.

However, it is limited to cases where values are **equispaced**, there are an **even number of segments and odd number of points**.



$$h = \frac{b-a}{n} \quad \text{n = \# of seg.} \quad a = x_0 \quad b = x_n$$

$$I = \int_{x_0}^{x_2} f^0(x)dx + \int_{x_2}^{x_4} f^2(x)dx + \cdots + \int_{x_{n-2}}^{x_n} f^{n-2}(x)dx$$

$$I = \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2) \right)$$

$$+ \frac{h}{3}\left(f(x_2) + 4f(x_3) + f(x_4) \right) \quad +$$

$$\cdots \quad + \frac{h}{3}\left(f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right)$$

$$I = \frac{h}{3}\left(f(x_0) + 4\sum_{i=1,3,5...}^{n-1} f(x_i) + 2\sum_{j=2,4,6...}^{n-2} f(x_j) + f(x_n) \right)$$

Simpson's 3/8 Rule

Fit a 3rd order Lagrange interpolating polynomial to four points and integrate

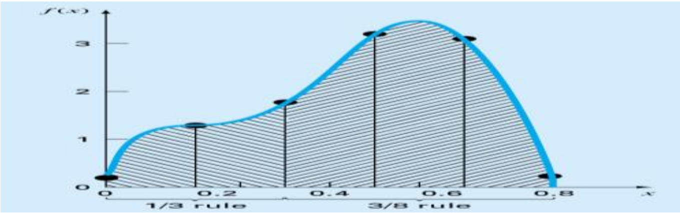
$$I = \int_a^b f(x)dx \cong \int_a^b f_3(x)dx$$

$$I \cong \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3) \right]$$

$$h = \frac{(b-a)}{3}$$

$$I \cong (b-a)\frac{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)}{8}$$

Simpson's 1/3 and 3/8 rules can be
applied in tandem to handle multiple applications with odd number of intervals



Example 1: Simpson's 1/3 rule

Evaluate $\int_0^6 \frac{dx}{1+x^2}$ **by dividing the interval into 6 equal subintervals**

.

Solution: Divide the interval (0,6) into six part of width h=1. The values of the function $f(x)$ are tabulated.

x (sec)	0	1	2	3	4	5	6
f(x)	1 y_0	0.5 y_1	0.2 y_2	0.1 y_3	0.0588 y_4	0.0385 y_5	0.027 y_6

by Simpson's 1/3 rule

$$\int_a^b f(x)dx = \frac{h}{3}\begin{pmatrix} f(x_0)+4\{f(x_1)+f(x_3)+f(x_5)\ldots\ldots\ldots f(x_{2n-1})\} \\ +2\{f(x_2)+f(x_4)+f(x_6)+f(x_8)\ldots\ldots\ldots f(x_{2n})\}+f(x_{2n}) \end{pmatrix}$$

$$\int_0^6 \frac{dx}{1+x^2} = \frac{1}{3}\left(1+4\{0.5+0.1+0.0385)\}+2\{0.2+0.0588)\}+0.027\right)=1.3662 \text{ units}$$

Example 2 (Simpson's 1/3 rule

The velocity of a particle which starts from rest is given by the following table.

t (sec)	0	2	4	6	8	10	12	14	16	28	20
v (ft/sec)	0	16	29	40	46	51	32	18	8	3	0

Evaluate using Simpson's *1/3* rule, the total distance travelled in 20 seconds.

Solution: From the definition, we have $v = \frac{ds}{dt}$, or $s = \int vdt$

Starting from rest, the distance travelled in 20 seconds is $s = \int_0^{20} vdt$

The step length is $h = 2$. Using Simpson's rule

$$\int_a^b f(x)dx = \frac{h}{3}\begin{pmatrix} f(x_0) + 4\{f(x_1) + f(x_3) + f(x_5)\ldots\ldots\ldots f(x_{2n-1})\} \\ +2\{f(x_2) + f(x_4) + f(x_6) + f(x_8)\ldots\ldots\ldots f(x_{2n})\} + f(x_{2n}) \end{pmatrix}$$

$$s = \int_0^{30} vdt = \frac{h}{2}\begin{pmatrix} f(0) + 4\{f(2) + f(6) + f(10) + f(14) + f(18)\} \\ +\{f(4) + f(8) + f(12) + f(16)\} + f(20) \end{pmatrix}$$

$= 0 + 4\{16 + 40 + 51 + 18 + 3\} + 2\{29 + 46 + 32 + 8\} + 0 = 494.667$ feet