**UNIT-II**

## Basic Terms and Definitions:

| | |
|---|---|
| Polynomial function | A function f(x) is said to be a polynomial function<br>If f(x) is a polynomial in x<br>ie.f(x)=$a_0x^n + a_1x^{n-1} + \cdots \ldots \ldots . . + a_{n-1}x + a_n$<br>Where $a_0 \neq 0$, the coefficients $a_o, a_1, \ldots \ldots . . a_n$ are real constants and n is a non-negative integer |
| Algebraic function | A function which is a sum (or) difference (or) product of two polynomials is called an algebraic function or polynomial function |
| **Transcendental function** | An equation which involves the non- polynomial functions or non-algebraic functions or transcendental functions such as trigonometric, logarithmic ,exponential functions and their combinations,<br><br>Ex: x sinx=1<br><br>x$e^x$+2=0 |
| Root of an equation | A number α is called a root of an equation f(x)=0 if f(α)=0. We also say that α is a zero of the function<br>Note: The roots of an equation are the abscissae of the points where the graph y= f( x) cuts the x-axis |
| Intermediate value theorem | Let f(x)=0 be the given equation.If f(x) is continuous in [a,b] and f(a),f(b) are having opposite signs,then the equation f(x)=0 have atleast one root lies between x=a, x=b |
| Regula -falsi method ( or) False position method | $x_{i+1} = \dfrac{x_{i-1}f(x_i) - x_i f(x_{i-1})}{f(x_i) - f x_{i-1})}$<br>For i= 0,1,2,3,………. |
| Newton- Raphson method (or)Newton's method of tangents (or) Newton's iteration method | $x_{i+1} = x_i - \dfrac{f(x_i)}{f^I(x_i)}$<br>F0r i= 0,1,2,3,……….. |
| Newton's iterative formula for finding reciprocal of a number | $\boldsymbol{x_{i+1} = 2x_i - Nx_i^2}$<br>F0r i= 0,1,2,3,……….. |
| Newton's iterative formula for finding square root of a number | $x_{i+1} = \dfrac{1}{2}\left[x_i + \dfrac{N}{x_i}\right]$<br>F0r i= 0,1,2,3,……….. |
| Newton's iterative formula for finding cube root of a number | $x_{i+1} = \dfrac{1}{3}\left[2x_i + \dfrac{N}{x_i^2}\right]$<br>F0r i= 0,1,2,3,……….. |

CONCEPTS

## Methods to find the roots of $f(x) = 0$

**Direct method**:

We know the solution of the polynomial equations such as linear equation $ax + b = 0$, and quadratic equation $ax^2 + bx + c = 0$, using direct methods or analytical methods. Analytical methods for the solution of cubic and quadratic equations are also available.

**Bisection method**: Bisection method is a simple iteration method to solve an equation. This method is also known as Bolzono method of successive bisection. Some times it is referred to as half-interval method. Suppose we know an equation of the form $f(x) = 0$ has exactly one real root between two real numbers $x_0, x_1$ .The number is choosen such that $f(x_0)$ and $f(x_1)$ will have opposite sign. Let us bisect the interval $[x_0, x_1]$ into two half intervals and find the mid point $x_2 = \frac{x_0 + x_1}{2}$ . If $f(x_2) = 0$ then $x_2$ is a root. If $f(x_1)$ and $f(x_2)$ have same sign then the root lies between $x_0$ and $x_2$. The interval is taken as $[x_0, x_2]$. Otherwise the root lies in the interval $[x_2, x_1]$.

**PROBLEMS**

**1). Find a root of the equation $x^3 - 5x + 1 = 0$ using the bisection method in 5 – stages**

Sol Let $f(x) = x^3 - 5x + 1$. We note that $f(0) > 0$ *and* $f(1) < 0$

$\therefore$ One root lies between 0 and 1

Consider $x_0 = 0$ *and* $x_1 = 1$

By Bisection method the next approximation is

$$x_2 = \frac{x_0 + x_1}{2} = \frac{1}{2}(0+1) = 0.5$$

$$\Rightarrow f(x_2) = f(0:5) = -1.375 < 0 \text{ and } f(0) > 0$$

We have the root lies between 0 and 0.5

Now $x_3 = \frac{0 + 0.5}{2} = 0.25$

We find $f(x_3) = -0.234375 < 0$ *and* $f(0) > 0$

Since $f(0) > 0$, we conclude that root lies between $x_0$ *and* $x_3$

The third approximation of the root is

$$x_4 = \frac{x_0 + x_3}{2} = \frac{1}{2}(0 + 0.25) = 0.125$$

We have $f(x_4) = 0.37495 > 0$

Since $f(x_4) > 0$ *and* $f(x_3) < 0$, the root lies between

$x_4 = 0.125 \quad and \quad x_3 = 0.25$

Considering the 4th approximation of the roots

$$x_5 = \frac{x_3 + x_4}{2} = \frac{1}{2}(0.125 + 0.25) = 0.1875$$

$f(x_5) = 0.06910 > 0$, since $f(x_5) > 0 \ and \ f(x_3) < 0$ the root must lie between

$$x_5 = 0.18758 \ and \ x_3 = 0.25$$

Here the fifth approximation of the root is

$$x_6 = \frac{1}{2}(x_5 + x_3)$$
$$= \frac{1}{2}(0.1875 + 0.25)$$
$$= 0.21875$$



We are asked to do up to 5 stages

We stop here 0.21875 is taken as an approximate value of the root and it lies between 0 and 1

**2) Find a root of the equation $x^3 - 4x - 9 = 0$ using bisection method in four stages**

Sol Let $f(x) = x^3 - 4x - 9$

We note that $f(2) < 0$ and $f(3) > 0$

$\therefore$ One root lies between 2 and 3

Consider $x_0 = 2 \ and \ x_1 = 3$

By Bisection method $x_2 = \frac{x_0 + x_1}{2} = 2.5$

Calculating $f(x_2) = f(2.5) = -3.375 < 0$

The root lies between $x_2 \ and \ x_1$

The second approximation is $x_3 = \frac{1}{2}(x_1 + x_2) = \frac{2.5+3}{2} = 2.75$

Now $f(x_3) = f(2.75) = 0.7969 > 0$

$\therefore$ The root lies between $x_2 \ and \ x_3$

Thus the third approximation to the root is

$$x_4 = \frac{1}{2}(x_2 + x_3) = 2.625$$

Again $f(x_4) = f(2.625) = -1.421 < 0$

$\therefore$ The root lies between $x_3 \ and \ x_4$

Fourth approximation is $x_5 = \frac{1}{2}(x_3 + x_4) = \frac{1}{2}(2.75 + 2.625) = 2.6875$

**False Position Method ( Regula – Falsi Method)**

In the false position method we will find the root of the equation $f(x)=0$ Consider two initial approximate values $x_0$ *and* $x_1$ near the required root so that $f(x_0)$ *and* $f(x_1)$ have different signs. This implies that a root lies between $x_0$ *and* $x_1$. The curve $f(x)$ crosses x-axis only once at the

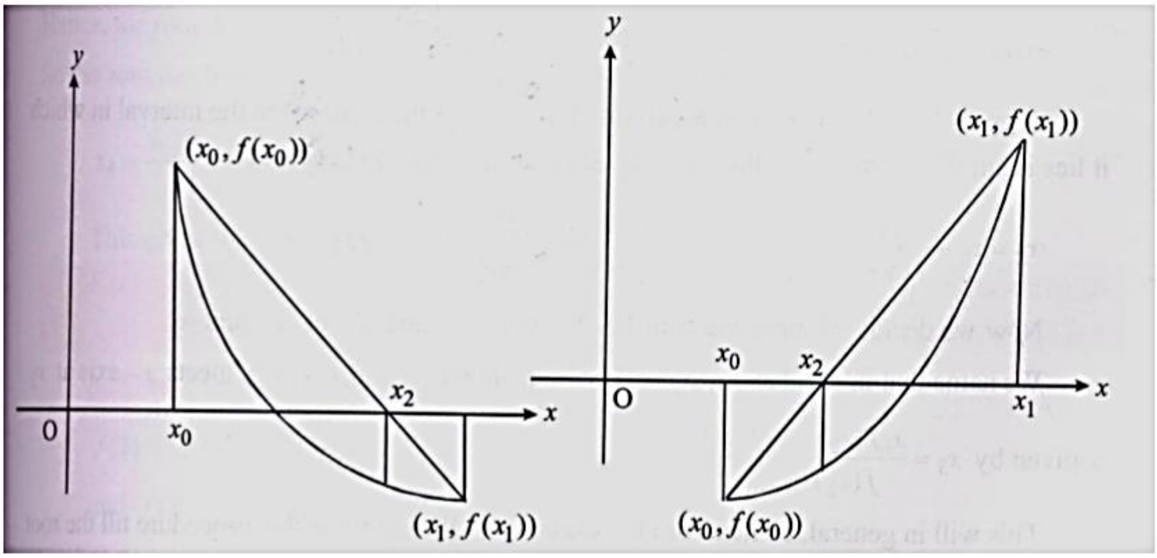
point $x_2$ lying between the points $x_0$ $and x_1$. Consider the point $A=(x_0, f(x_0))$ and $B=(x_1, f(x_1))$

on the graph and suppose they are connected by a straight line. Suppose this line cuts x-axis at$x_2$. We calculate the value of $f(x_2)$ at the point. If $f(x_0)$ *and* $f(x_2)$ are of opposite signs, then the root lies between $x_0$ *and* $x_2$ and value $x_1$ is replaced by $x_2$

Other wise the root lies between $x_2$ and $x_1$ and the value of $x_0$ is replaced by$x_2$.

Another line is drawn by connecting the newly obtained pair of values. 

Again the point here cuts the x-axis is a closer approximation to the root. This process is repeated as many times as required to obtain the desired accuracy. It can be observed that the points $x_2, x_3, x_4, \ldots$obtained converge to the expected root of the equation $y=f(x)$



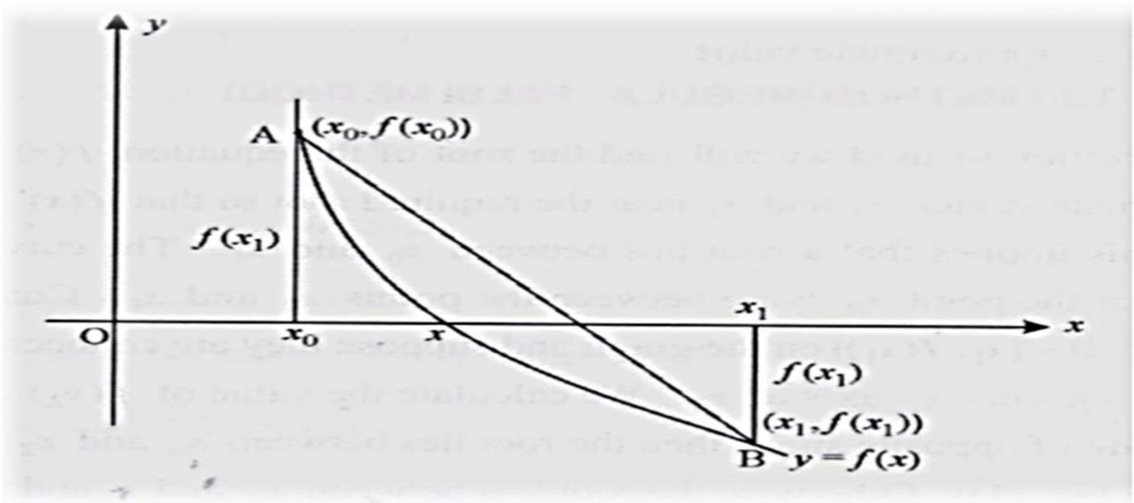
**To Obtain the equation to find the next approximation to the root**

Let $A=\left(x_0, f\left(x_0\right)\right)$ *and* $B=\left(x_1, f\left(x_1\right)\right)$ be the points on the curve $y=f\left(x\right)$ Then the equation to the chord AB is $\frac{y-f(x_0)}{x-x_0}=\frac{f(x_1)-f(x_0)}{x_1-x_0}$ ------(1)

At the point C where the line AB crosses the x – axis, where $f(x) = 0$ *ie*, $y = 0$

From (1), we get $x=x_0-\dfrac{x_1-x_0}{f\left(x_1\right)-f\left(x_0\right)}f\left(x_0\right)$ --------------(2)

x is given by (2) serves as an approximated value of the root, when the interval in which it lies is small. If the new value of x is taken as $x_2$ then (2) becomes




$$x_2=x_0-\frac{\left(x_1-x_0\right)}{f\left(x_1\right)-f\left(x_0\right)}f\left(x_0\right)$$

$$=\frac{x_0 f\left(x_1\right)-x_1 f\left(x_0\right)}{f\left(x_1\right)-f\left(x_0\right)} \quad \text{--------------(3)}$$

Now we decide whether the root lies between

$x_0$ *and* $x_2$ $(or)$ $x_2$ *and* $x_1$

We name that interval as $(x_1, x_2)$ The line joining $(x_1, y_1), (x_2, y_2)$ meets x – axis at $x_3$ is given by $x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$

This will in general, be nearest to the exact root. We continue this procedure till the root is found to the desired accuracy

The iteration process based on (3) is known as the method of false position

The successive intervals where the root lies, in the above procedure are named as $(x_0, x_1), (x_1, x_2), (x_2, x_3)$ etc

Where $x_i < x_{i+1}$ and $f(x_0), f(x_{i+1})$ are of opposite signs.



Also $x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$

**PROBLEMS:**

1. **By using Regula - Falsi method, find an approximate root of the equation** $x^4 - x - 10 = 0$ **that lies between 1.8 and 2. Carry out three approximations**

Sol.Let us take $f(x) = x^4 - x - 10$ and $x_0 = 1.8, x_1 = 2$

Then $f(x_0) = f(1.8) = -1.3 < 0$ and $f(x_1) = f(2) = 4 > 0$

Since $f(x_0)$ and $f(x_1)$ are of opposite signs,the equation $f(x) = 0$ has a root between $x_0$ *and* $x_1$

The first order approximation of this root is

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$



$$= 1.8 - \frac{2 - 1.8}{4 + 1.3} \times (-1.3)$$

$$= 1.849$$

We find that $f(x_2) = -0.161$ so that $f(x_2)$ *and* $f(x_1)$ are of opposite signs. Hence the root lies between $x_2$ *and* $x_1$ and the second order approximation of the root is

$$x_3 = x_2 - \left[\frac{x_1 - x_2}{f(x_1) - f(x_2)}\right] . f(x_2)$$

$$= 1.8490 - \left[\frac{2 - 1.849}{0.159}\right] \times (-0.159)$$

$$= 1.8548$$



We find that $f(x_3) = f(1.8548)$

$$= -0.019$$

So that $f(x_3)$ *and* $f(x_2)$ are of the same sign. Hence, the root does not lie between $x_2$ *and* $x_3$. But $f(x_3)$ *and* $f(x_1)$ are of opposite signs. So the root lies between $x_3$ *and* $x_1$ and the third order approximate value of the root is $x_4 = x_3 - \left[\frac{x_1 - x_3}{f(x_1) - f(x_3)}\right] f(x_3)$

$$= 1.8548 - \frac{2 - 1.8548}{4 + 0.019} \times (-0.019)$$

$= 1.8557$

This gives the approximate value of x.

**2. Find out the roots of the equation $x^3 - x - 4 = 0$ using False position method**

Sol. Let $f(x) = x^3 - x - 4 = 0$

Then $f(0) = -4, f(1) = -4, f(2) = 2$

Since $f(1)$ *and* $f(2)$ have opposite signs the root lies between 1 and 2

By False position method $x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$

$$x_2 = \frac{(1 \times 2) - 2(-4)}{2 - (-4)}$$

$$= \frac{2+8}{6} = \frac{10}{6} = 1.666$$

$$f(1.666) = (1.666)^3 - 1.666 - 4$$
$$= -1.042$$

Now, the root lies between 1.666 and 2

$$x_3 = \frac{1.666 \times 2 - 2 \times (-1.042)}{2 - (-1.042)} = 1.780$$

$$f(1.780) = (1.780)^3 - 1.780 - 4$$
$$= -0.1402$$

Now, the root lies between 1.780 and 2

$$x_4 = \frac{1.780 \times 2 - 2 \times (-0.1402)}{2 - (-0.1402)} = 1.794$$

$$f(1.794) = (1.794)^3 - 1.794 - 4$$
$$= -0.0201$$

Now, the root lies between 1.794 and 2

$$x_5 = \frac{1.794 \times 2 - 2 \times (-0.0201)}{2 - (-0.0201)} = 1.796$$

$$f(1.796) = (1.796)^3 - 1.796 - 4 = -0.0027$$

Now, the root lies between 1.796 and 2

$$x_6 = \frac{1.796\times 2 - 2\times(-0.0027)}{2-(-0.0027)} = 1.796 \qquad \text{The root is } 1.796$$

**<u>Newton- Raphson Method:-</u>**

The Newton- Raphson method is a powerful and elegant method to find the root of an equation. This method is generally used to improve the results obtained by the previous methods.

Let $x_0$ be an approximate root of $f(x)=0$ and let $x_1 = x_0 + h$ be the correct root which implies that $f(x_1)=0$. We use Taylor's theorem and expand $f(x_1)=f(x_0+h)=0$

$$\Rightarrow f(x_0)+hf^1(x_0)=0$$

$$\Rightarrow h = -\frac{f(x_0)}{f^1(x_0)}$$

Substituting this in $x_1$ ,we get

$$x_1 = x_0 + h$$

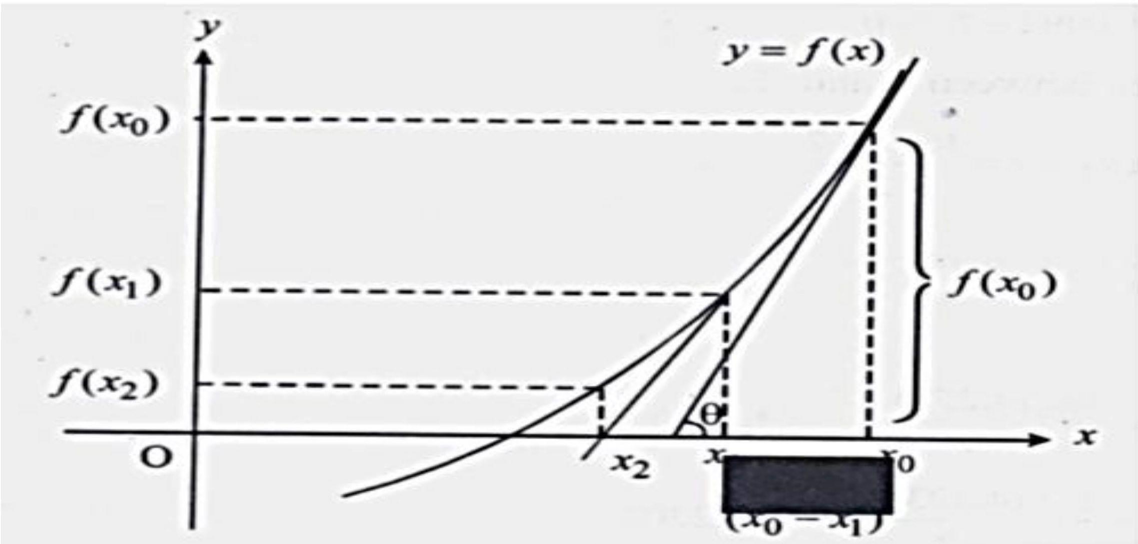


$$= x_0 - \frac{f(x_0)}{f^1(x_0)}$$

$\therefore x_1$ is a better approximation than $x_0$

Successive approximations are given by

$x_2, x_3 \ldots\ldots\ldots x_{n+1}$ where $x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)}$




**Problems:**

1. Apply Newton – Raphson method to find an approximate root, correct to three decimal places, of the equation $x^3-3x-5=0$, which lies near $x=2$

**<u>Sol:-</u>** Here $f(x)=x^3-3x-5=0$ *and* $f^1(x)=3(x^2-1)$

$\therefore$ The Newton – Raphson iterative formula

$$x_3 = \frac{2x_2^3 + 5}{3(x_2^3 - 1)} = \frac{2 \times (2.2806)^3 + 5}{3[(2.2806)^2 - 1]} = 2.2790$$

$$x_4 = \frac{2 \times (2.2790)^3 + 5}{3[(2.2790)^2 - 1]} = 2.2790$$

Since $x_3$ and $x_4$ are identical up to 3 places of decimal, we take $x_4 = 2.279$ as the required root, correct to three places of the decimal

Using Newton – Raphson method

a) Find square root of a number

b) Find reciprocal of a number

a) **Square root:-**



Let $f(x) = x^2 - N = 0$, where N is the number whose square root is to be found.



The solution to $f(x)$ is then $x = \sqrt{N}$

Here $f'(x) = 2x$

By Newton-Raphson technique

$$x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)} = x_i - \frac{x_i^2 - N}{2x_i}$$

$$\Rightarrow x_{i+1} = \frac{1}{2}\left[x_i + \frac{N}{x_i}\right]$$

Using the above iteration formula the square root of any number N can be found to any desired accuracy. For example, we will find the square root of $N = 24$.

Let the initial approximation be $x_0 = 4.8$



$$x_1 = \frac{1}{2}\left(4.8 + \frac{24}{4.8}\right) = \frac{1}{2}\left(\frac{23.04 + 24}{4.8}\right) = \frac{47.04}{9.6} = 4.9$$

$$x_2 = \frac{1}{2}\left(4.9 + \frac{24}{4.9}\right) = \frac{1}{2}\left(\frac{24.01 + 24}{4.9}\right) = \frac{48.01}{9.8} = 4.898$$

$$x_3 = \frac{1}{2}\left(4.898 + \frac{24}{4.898}\right) = \frac{1}{2}\left(\frac{23.9904 + 24}{4.898}\right) = \frac{47.9904}{9.796} = 4.898$$

Since $x_2 = x_3$, there fore the solution to $f(x) = x^2 - 24 = 0$ is $4.898$. That means,

the square root of 24 is 4.898

**b) Reciprocal:-**

Let $f(x) = \frac{1}{x} - N = 0$ where N is the number whose reciprocal is to be foun

The solution to $f(x)$ is then $x = \frac{1}{N}$. Also, $f^1(x) = \frac{-1}{x^2}$

To find the solution for $f(x) = 0$, apply Newton – Raphson method

$$x_{i+1} = x_i - \frac{\left(\frac{1}{x_i} - N\right)}{-1/x_i^2} = x_i(2 - x_i N)$$

For example, the calculation of reciprocal of 22 is as follows

Assume the initial approximation be $x_0 = 0.045$ 

For example, the calculation of reciprocal of 22 is as follows

Assume the initial approximation be $x_0 = 0.045$

$$\therefore x_1 = 0.045(2 - 0.045 \times 22)$$
$$= 0.045(2 - 0.99)$$
$$= 0.0454(1.01) = 0.0454$$
$$x_2 = 0.0454(2 - 0.0454 \times 22)$$
$$= 0.0454(2 - 0.9988)$$
$$= 0.0454(1.0012) = 0.04545$$
$$x_3 = 0.04545(2 - 0.04545 \times 22)$$
$$= 0.04545(1.0001) = 0.04545$$

$$x_4 = 0.04545(2 - 0.04545 \times 22)$$
$$= 0.04545(2 - 0.99998)$$
$$= 0.04545(1.00002)$$

=0.0454509

**Crout's Triangularisation method (or)LU-Decomposition method (or)**

## Method of Factorization (Triangularisation):

Triangular Decomposition Method:

This method is based on the fact that a square matrix A can be factorized into the form LU, where L is the unit lower triangular matrix and U is the upper triangular matrix.

Consider the linear system

$a_{11} x_1+a_{12}x_2+a_{13}x_3 =b_1$

$a_{21} x_1+a_{22}x_2+a_{23}x_3 =b_2$

$a_{31} x_1+a_{32}x_2+a_{33}x_3 =b3$

which can be written in the form Ax = B ---------(1)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22}. & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = L.\ U \text{ -----(2) and } X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\text{Where } L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Then (1) → LUX = B.

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$$\text{If we put } UX = Y \text{ where } Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Then (2) becomes LY = B

The system is equals to $y_1 = b_1$

$$\left.\begin{aligned} y_1 &= b_1 \\ l_{21}+y_2 &= b_2 \\ l_{31}y_1+l_{32}y_2+y_3 &= b_3 \end{aligned}\right\} \quad (3)$$

here $y_1,y_2,y_3$ are solved by forward substitution using (3) we get

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$u_{11} x_1+u_{12}x_2+u_{13}x_3 =y_1$

$u_{22}x_2+u_{23}x_3 \quad =y_2$

$u_{33}x_3 \quad =y_3$

from these we can sole for $x_1,x_2,x_3$ by backward substitution .

The method of computing L and U is outlined below from (2) * we get

$$\begin{bmatrix}1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{bmatrix}\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22}. & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$$

Equating corresponding elements, we get

$u_{11} = a_{11}$ $\quad$ $l_{21}\, u_{11} = a_{21} \Rightarrow l_{21} = a_{21}/a_{11}$ and

$u_{12} = a_{12}$ $\quad$ $l_{31}\, u_{11} = a_{31} \Rightarrow l_{31} = a_{31}/a_{11}$

$u_{13} = a_{13}$ $\quad$ $l_{21}\, u_{12} + u_{22} = a_{22} \Rightarrow l_{21}a_{12} + u_{22} = a_{22} \Rightarrow u_{22} = a_{22} - a_{21}a_{12}/a_{11}$

$l_{31}u_{12}+l_{32}u_{22} = a_{32} \Rightarrow l_{32} = [a_{32}-l_{31}\, u_{12}] / u_{22}$ and

$l_{31}a_{13}+l_{32}a_{23}+u_{33} = a_{33}$ from which $u_{33}$ can be calculated.

Ex : Solve the following system by the method of factorization x+3y+8z =4, x+4y+3z = -2, x+3y+4z =1

**Soln:** The given system can write AX = B;

$$A \Rightarrow \begin{bmatrix}1 & 3 & 8\\ 1 & 4 & 3\\ 1 & 3 & 4\end{bmatrix} \quad X = \begin{bmatrix}x\\ y\\ z\end{bmatrix} \text{ and } B = \begin{bmatrix}4\\ -2\\ 1\end{bmatrix}$$

Let A = LU

$$\text{Where } L = \begin{bmatrix}1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{bmatrix} \text{ and } U = \begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}u_{11} & u_{12} & u_{13}\\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23}\\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33}\end{bmatrix} = \begin{bmatrix}1 & 3 & 8\\ 1 & 4 & 3\\ 1 & 3 & 4\end{bmatrix}$$

➔ $u_{11} = 1,\ u_{12} = 3,\ u_{13} = 8$

$l_{21} u_{11} = 1$ ➔ $l_{21} = 1$ and $l_{31}u_{11} = 1$ ➔ $l_{31} = 1$

from the equations $l_{21}u_{12}+u_{22} = 4$ and

$l_{21}u_{13}+u_{23} = 3$, we get

$u_{22} = 4-l_{21}u_{12}$

$= 4-3 = 1$

$u_{23} = 3-l_{21}u_{13}$

$= 3-8 = -5$

By using $l_{31}u_{12}+l_{32}u_{22} = 3$ and $l_{31}u_{13}+l_{32}u_{23}+u_{33} = 4$ we get

$l_{33} = 4-l_{31}u_{13}-l_{32}u_{23}$

$= 4-1(8)-(0)5 = -4$

Thus $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 3 & 8 \\ 0 & 1 & -5 \\ 0 & 0 & -4 \end{bmatrix}$

Let UX = Y where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ then LY = B

➔ $\begin{bmatrix} 1 & 3 & 8 \\ 0 & 1 & -5 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ ---------(1)

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}4\\-2\\1\end{bmatrix} \text{-----(2)}$$

From (2) $y_1= 4$, $y_2= -2$ and $y_1+y_3 =1$

$y_3= -3$

from (1)

$x = -29/4$ , $y =7/4$ and $z= 3/4$

-SEIDAL GAUSS ITERATION METHOD (OR) METHOD OF SUCCESSIVE DISPLACEMENT

Consider the system of equations

$$a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n = b_n$$

Has a unique solution.

The coefficient matrix $A$ has no zeros on its main diagonal, namely, $a_{11}, a_{22}, \dots, a_{nn}$ are nonzeros.

To begin, solve the 1st equation for $x_1$, the 2nd equation for $x_2$ and so on to obtain the rewritten equations:

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots a_{1n}x_n)$$
$$x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots a_{2n}x_n)$$
$$\vdots$$
$$x_n = \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots a_{n,n-1}x_{n-1})$$

Then make an initial guess of the solution $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \dots x_n^{(0)})$. Substitute these values into the right hand side the of the rewritten equations to obtain the *first approximation*, $\left(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \dots x_n^{(1)}\right)$.

This accomplishes one **iteration**.

In the same way, the *second approximation* $\left(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \dots x_n^{(2)}\right)$ is computed by substituting the first approximation's $x$-vales into the right hand side of the rewritten equations.

This process is continued as long as any two approximate values are approximately equal

Therefore the solution is $X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$

Use the Gauss –Seidal iteration method to approximate the solution of following system of linear equations (or)

Use the Jacobi method to approximate the solution of the following system of linear equations.

$$5x_1 - 2x_2 + 3x_3 = -1$$
$$-3x_1 + 9x_2 + x_3 = 2$$
$$2x_1 - x_2 - 7x_3 = 3$$

Continue the iterations until two successive approximations are identical when rounded to three significant digits.

**Solution** To begin, write the system in the form

$$x_1 = -\frac{1}{5} + \frac{2}{5}x_2 - \frac{3}{5}x_3$$
$$x_2 = \frac{2}{9} + \frac{3}{9}x_1 - \frac{1}{9}x_3$$
$$x_3 = -\frac{3}{7} + \frac{2}{7}x_1 - \frac{1}{7}x_2.$$

Because you do not know the actual solution, choose

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0$$ Initial approximation

as a convenient initial approximation. So, the first approximation is

$$x_1 = -\frac{1}{5} + \frac{2}{5}(0) - \frac{3}{5}(0) = -0.200$$
$$x_2 = \frac{2}{9} + \frac{3}{9}(0) - \frac{1}{9}(0) \approx 0.222$$
$$x_3 = -\frac{3}{7} + \frac{2}{7}(0) - \frac{1}{7}(0) \approx -0.429.$$

Continuing this procedure, you obtain the sequence of approximations shown in Table 10.1.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $x_1$ | 0.000 | −0.200 | 0.146 | 0.192 | 0.181 | 0.185 | 0.186 | 0.186 |
| $x_2$ | 0.000 | 0.222 | 0.203 | 0.328 | 0.332 | 0.329 | 0.331 | 0.331 |
| $x_3$ | 0.000 | −0.429 | −0.517 | −0.416 | −0.421 | −0.424 | −0.423 | −0.423 |

Because the last two columns in Table are identical, you can conclude that to three significant digits the solution is

$x_1=0.186 \quad x_2=0.331 \quad x_3=-0.423$

IMPORTANT QUESTIONS

1. Define Algebraic and Transcendental equations and give an examples
2.Find a root of the equation $x^3 - 5x + 1 = 0$ using the bisection method in 5 stages. **(JNTUA 2010)**
3. By using bisection method, find an approximate root of the equation $\sin x = \frac{1}{x}$ that lies between $x = 1$ *and* $x = 1.5$(measured in radians).Carry out computation upto $7^{th}$ stage. **(JNTUA 2010, (JNTUH 2009))**
4. Find a real root of the equation $x \log_{10} x = 1.2$ which lies between 2 *and* 3 by bisection method in 5 *stages* **(JNTUA 2008)**
5. Find a real root of the equation $x \log_{10} x = 1.2$ using False position method. **(JNTUA 2005,08,09,10,11,12, (JNTUK 2008))**
6. By using Regula-Falsi method , find an approximate root of the equation $x^4 - x - 10 = 0$ that lies between 1.8 *and* 2.Carry out three approximations. **(JNTUA 2010)**
7. Determine the root of $xe^x - 2 = 0$ by the method of falsi position **(JNTUA 2015)**
8. Determine the root of the equation $x^2 - 3x + 1 = 0$ using regula-falsi method upto 3

stages**(JNTUA 2016)**

9. Find a real root for $x\tan x+1=0$ Using Newton-Raphson method. **(JNTUA 2006,2011)**

10. Find a root of $e^x\sin x=1$ Using Newton-Raphson method. **(JNTUA 2006,10)**

11. ) Develop an algorithm using Newton-Raphson method to find square root of a positive number N ) **(JNTUA 2015)**

12. Using Newton-Raphson method

a)Find square root of a number and hence compute the value of $\sqrt{25}$ **(JNTUA 2004)**

b)Find Reciprocal of a number and hence compute the value of $\frac{1}{26}$.

c) Find cube root of a number and hence compute the value of $\sqrt[3]{12}$)

13. Find a root of $e^x\sin x=1$ Using Newton-Raphson method. **(JNTUA 2006,10)**

14. Using Crout's triangularisation method to solve the following equations

$x+3y+8z=4,\ x+4y+3z=-2\ ,x+3y+4z=1.$ $\left(x=\frac{19}{4},y=\frac{-9}{4},\ z=\frac{3}{4}\right)$ **(JNTUA 2015)**

15. Solve the following system by the method of triangularisation

$2x-3y+10z=3,-x+4y+2z=20\ ,5x+2y+z=-12\ .$ $(x=-4,y=3,\ z=2)$

16. Solve the following system of equations by gauss-seidal method

i) Start with $(2\ ,2,-1)$ *and solve* $5x_1-x_2+x_3=10,\ 2x_1+4x_2=12,\ x_1+x_2+5x_3=-1$

ii) $8x_1+x_2-x_3=8,\ 2x_1+x_2+9x_3=12,\ x_1-7x_2+2x_3=-4$