UNIT-4

TRACKING WITH RADAR

Single Target Tracking

Tracking radar systems are used to measure the target's relative position in range, azimuth angle, elevation angle, and velocity. Then, by using and keeping track of these measured parameters the radar can predict their future values. Target tracking is important to military radars as well as to most civilian radars. In military radars, tracking is responsible for fire control and missile guidance; in fact, missile guidance is almost impossible without proper target tracking. Commercial radar systems, such as civilian airport traffic control radars, may utilize tracking as a means of controlling incoming and departing airplanes.

Tracking techniques can be divided into range/velocity tracking and angle tracking. It is also customary to distinguish between continuous single-target tracking radars and multi-target track-while-scan (TWS) radars. Tracking radars utilize pencil beam (very narrow) antenna patterns. It is for this reason that a separate search radar is needed to facilitate target acquisition by the tracker. Still, the tracking radar has to search the volume where the target's presence is suspected. For this purpose, tracking radars use special search patterns, such as helical, T.V. raster, cluster, and spiral patterns, to name a few.

Angle Tracking

Angle tracking is concerned with generating continuous measurements of the target's angular position in the azimuth and elevation coordinates. The accuracy of early generation angle tracking radars depended heavily on the size of the pencil beam employed. Most modern radar systems achieve very fine angular measurements by utilizing monopulse tracking techniques.

Tracking radars use the angular deviation from the antenna main axis of the target within the beam to generate an error signal. This deviation is normally measured from the antenna's main axis. The resultant error signal describes how much the target has deviated from the beam main axis. Then, the beam position is continuously changed in an attempt to produce a zero error signal. If the radar beam is normal to the target (maximum gain), then the target angular position would be the same as that of the beam. In practice, this is rarely the case.

In order to be able to quickly achieve changing the beam position, the error signal needs to be a linear function of the deviation angle. It can be shown that this condition requires the beam's axis to be squinted by some angle (squint angle) off the antenna's main axis.

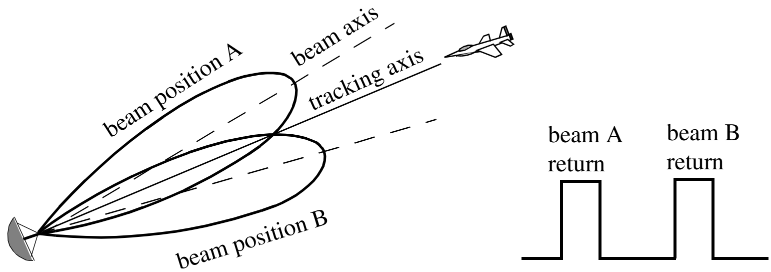
Sequential Lobing

Sequential lobing is one of the first tracking techniques that was utilized by the early generation of radar systems. Sequential lobing is often referred to as lobe switching or sequential switching. It has a tracking accuracy that is limited by the pencil beam width used and by the noise caused by either mechanical or electronic switching mechanisms. However, it is very simple to implement. The pencil beam used in sequential lobing must be symmetrical (equal azimuth and elevation beam widths).

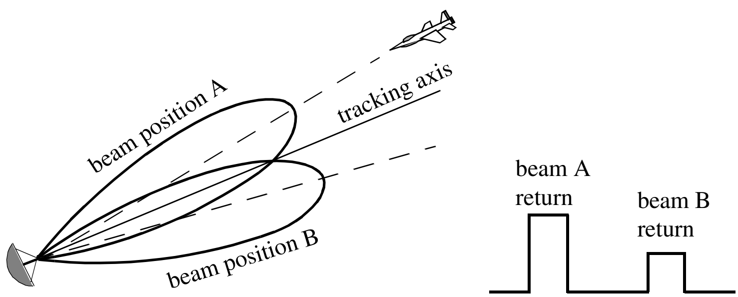
Tracking is achieved (in one coordinate) by continuously switching the pen- cil beam between two pre-determined symmetrical positions around the antenna's Line of Sight (LOS) axis. Hence, the name sequential lobing is adopted. The LOS is called the radar tracking axis, as illustrated in fig.

As the beam is switched between the two positions, the radar measures the returned signal levels. The difference between the two measured signal levels is used to compute the angular error signal. For example, when the target is tracked on the tracking axis, as the case in fig, the voltage difference is zero and, hence, is also the error signal. However, when the target is off the tracking axis, as in fig, a nonzero error signal is produced. The sign of the voltage difference determines the direction in which the antenna must be moved. Keep in mind, the goal here is to make the voltage difference be equal to zero.

In order to obtain the angular error in the orthogonal coordinate, two more switching positions are required for that coordinate. Thus, tracking in two coordinates can be accomplished by using a cluster of four antennas (two for each coordinate) or by a cluster of five antennas. In the latter case, the middle antenna is used to transmit, while the other four are used to receive.



(a)



(b)

Figure . Sequential lobing. (a) Target is located on track axis.
(b) Target is off track axis.

Conical Scan

Conical scan is a logical extension of sequential lobing where, in this case, the antenna is continuously rotated at an offset angle, or has a feed that is rotated about the antenna's main axis. fig shows a typical conical scan beam. The beam scan frequency, in radians per second, ω_s is denoted as. The angle between the antenna's LOS and the rotation axis φ is the squint angle . The antenna's beam position is continuously changed so that the target will always be on the tracking axis.

fig shows a simplified conical scan radar system. The envelope detec- tor is used to extract the return signal amplitude and the Automatic Gain Con- trol (AGC) tries to hold the receiver output to a constant value. Since the AGC operates on large time constants, it can hold the average signal level constant and still preserve the signal rapid scan variation. It follows that the tracking error signals (azimuth and elevation) are functions of the target's RCS; they are functions of its angular position with the main beam axis.

In order to illustrate how conical scan tracking is achieved, we will first consider the case shown in fig. In this case, as the antenna rotates around the tracking axis all target returns have the same amplitude (zero error signal). Thus, no further action is required.

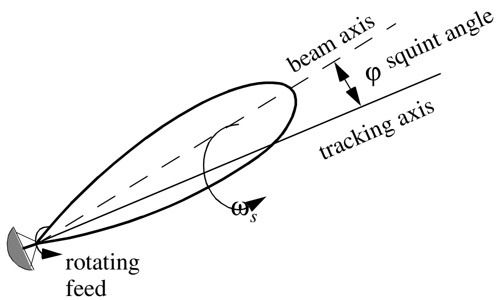


Figure . Conical scan beam.

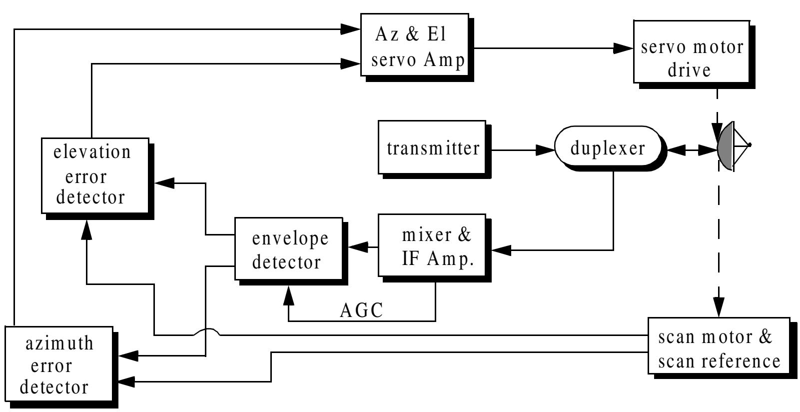


Figure . Simplified conical scan radar system.

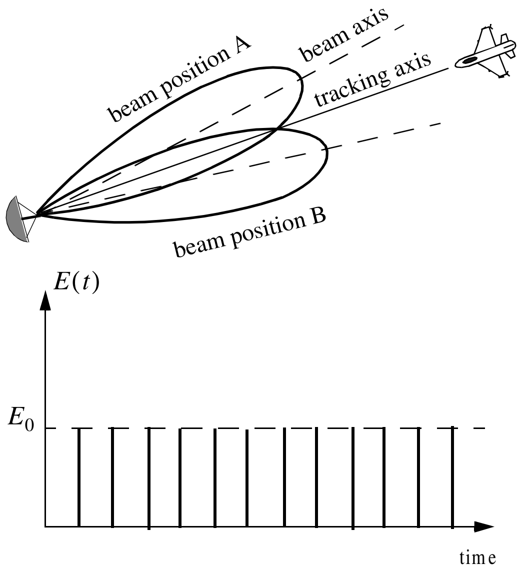


Figure . Error signal produced when the target is on the tracking axis for conical scan.

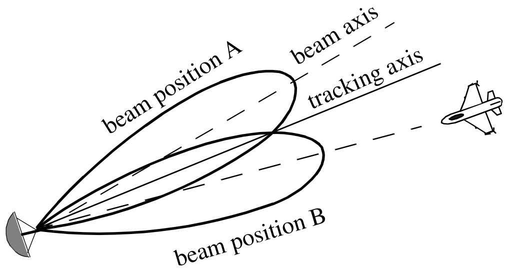
Next, consider the case depicted by fig Here, when the beam is at position B, returns from the target will have maximum amplitude. And when the antenna is at position A, returns from the target have minimum amplitude. Between those two positions, the amplitude of the target returns will vary between the maximum value at position B, and the minimum value at position A. In other words, Amplitude Modulation (AM) exists on top of the returned signal. This AM envelope corresponds to the relative position of the target within the beam. Thus, the extracted AM envelope can be used to derive a servo-control system in order to position the target on the tracking axis.

Now, let us derive the error signal expression that is used to drive the servo- control system. Consider the top view of the beam axis location shown in figure below $\varphi = 0$ Assume that is the starting beam position. The locations for maxi- mum and minimum target returns are also identified. The quantity ε defines the distance between the target location and the antenna's tracking axis. It fol- lows that the azimuth and elevation errors are, respectively, given by

$$\varepsilon_a = \varepsilon \sin\varphi \quad \textbf{(11.1)}$$

$$\varepsilon_e = \varepsilon \cos\varphi \quad \textbf{(11.2)}$$

These are the error signals that the radar uses to align the tracking axis on the target.



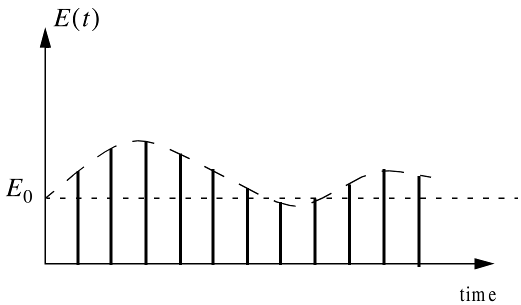


Figure . Error signal produced when the target is off the tracking axis for conical scan.

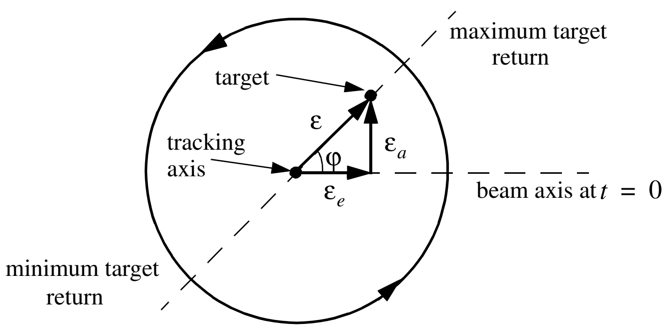


Figure . Top view of beam axis for a complete scan.

The AM signal $E(t)$ can then be written as

$$E(t) = E_0 \cos(\omega_s t - \varphi) = E_0 \varepsilon_e \cos \omega_s t + E_0 \varepsilon_a \sin \omega_s t$$

where E_0 is a constant called the error slope, ω_s is the scan frequency in radians per seconds, and φ is the angle already defined. The scan reference is the signal that the radar generates to keep track of the antenna's position around a complete path (scan). The elevation error signal is obtained by mixing the signal $E(t)$ with $\cos \omega_s t$ (the reference signal) followed by low pass filtering. More precisely,

$$E_e(t) = E_0 \cos(\omega_s t - \varphi) \cos \omega_s t = -\frac{1}{2} E_0 \cos \varphi + \frac{1}{2} \cos(2\omega_s t - \varphi) \tag{11.4}$$

and after low pass filtering we get

$$E_e(t) = -\frac{1}{2} E_0 \cos \varphi$$

Negative elevation error drives the antenna beam downward, while positive elevation error drives the antenna beam upward. Similarly, the azimuth error signal is obtained by multiplying $E(t)$ by $\sin \omega_s t$ followed by low pass filtering. It follows that

$$E_a(t) = \frac{1}{2} E_0 \sin \varphi$$

The antenna scan rate is limited by the scanning mechanism (mechanical or electronic), where electronic scanning is much faster and more accurate than mechanical scan. In either case, the radar needs at least four target returns to be able to determine the target azimuth and elevation coordinates (two returns per coordinate). Therefore, the maximum conical scan rate is equal to one fourth of the PRF. Rates as high as 30 scans per seconds are commonly used.

The conical scan squint angle needs to be large enough so that a good error signal can be measured. However, due to the squint angle, the antenna gain in the direction of the tracking axis is less than maximum. Thus, when the target is in track (located on the tracking axis), the SNR suffers a loss equal to the drop in the antenna gain. This loss is known as the squint or crossover loss. The squint angle is normally chosen such that the two-way (transmit and receive) crossover loss is less than a few decibels.

Amplitude Comparison Monopulse

Amplitude comparison monopulse tracking is similar to lobing in the sense that four squinted beams are required to measure the target's angular position. The difference is that the four beams are generated simultaneously rather than sequentially. For this purpose, a special antenna feed is utilized such that the four beams are produced using a single pulse, hence the name "monopulse." Additionally, monopulse tracking is more accurate and is not susceptible to lobing anomalies, such as AM jamming and gain inversion ECM. Finally, in sequential and conical lobing variations in the radar echoes degrade the tracking accuracy; however, this is not a problem for monopulse techniques since a single pulse is used to produce the error signals. Monopulse tracking radars can employ both antenna reflectors as well as phased array antennas.

Figure show a typical monopulse antenna pattern. The four beams A, B, C, and D represent the four conical scan beam positions. Four feeds, mainly horns, are used to produce the monopulse antenna pattern. Amplitude monopulse processing requires that the four signals have the same phase and different amplitudes.

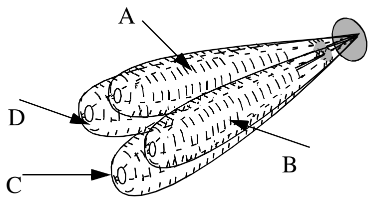


Figure . Monopulse antenna pattern.

A good way to explain the concept of amplitude monopulse technique is to represent the target echo signal by a circle centered at the antenna's tracking axis, as illustrated by Fig. 11.8a, where the four quadrants represent the four beams. In this case, the four horns receive an equal amount of energy, which indicates that the target is located on the antenna's tracking axis. However, when the target is off the tracking axis (Figs. 11.8b-d), an unbalance of energy occurs in the different beams. This unbalance of energy is used to generate an error signal that drives the servo-control system. Monopulse processing consists of computing a sum Σ and two difference Δ (azimuth and elevation) antenna patterns. Then by dividing a Δ channel voltage by the Σ channel voltage, the angle of the signal can be determined.

The radar continuously compares the amplitudes and phases of all beam returns to sense the amount of target displacement off the tracking axis. It is critical that the phases of the four signals be constant in both transmit and receive modes. For this purpose, either digital networks or microwave compar- ator circuitry are utilized. Fig. shows a block diagram for a typical micro- wave comparator, where the three receiver channels are declared as the sum channel, elevation angle difference channel, and azimuth angle difference channel.

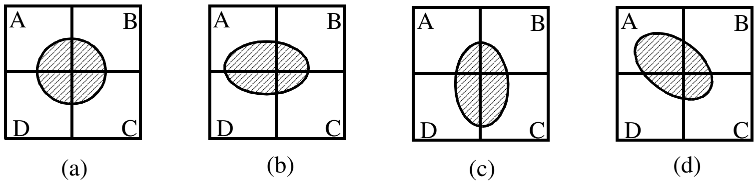


Figure Illustration of monopulse concept. (a) Target is on the tracking axis. (b) - (d) Target is off the tracking axis.

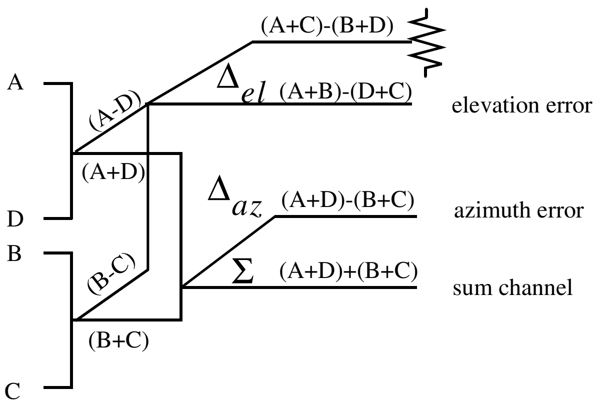


Figure . Monopulse comparator.

To generate the elevation difference beam, one can use the beam difference (A−D) or (B−C). However, by first forming the sum patterns (A+B) and (D+C) and then computing the difference (A+B)−(D+C), we achieve a stronger elevation difference signal, Δ_{el}. Similarly, by first forming the sum patterns (A+D) and (B+C) and then computing the difference (A+D)−(B+C), a stronger azimuth difference signal, Δ_{az}, is produced.

A simplified monopulse radar block diagram is shown in Fig. The sum channel is used for both transmit and receive. In the receiving mode the sum channel provides the phase reference for the other two difference chan- nels. Range measurements can also be obtained from the sum channel. In order to illustrate how the sum and difference antenna patterns are formed, we will assume a $\sin\varphi/\varphi$ single element antenna pattern and squint angle φ_0.

The sum signal in one coordinate (azimuth or elevation) is then given by

$$\Sigma(\varphi) = \frac{\sin(\varphi - \varphi_0)}{(\varphi - \varphi_0)} + \frac{\sin(\varphi + \varphi_0)}{(\varphi + \varphi_0)}$$

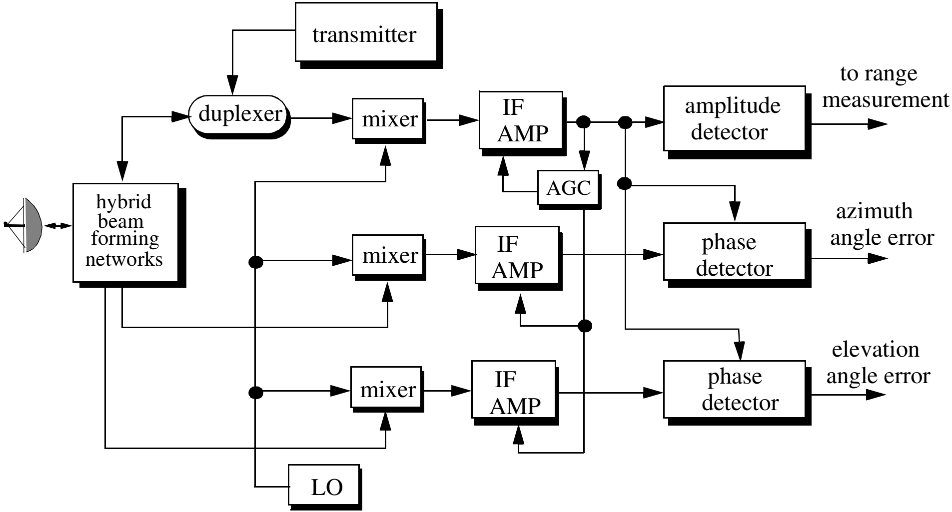


Figure . Simplified amplitude comparison monopulse radar block diagram.

Phase Comparison Monopulse

Phase comparison monopulse is similar to amplitude comparison monopulse in the sense that the target angular coordinates are extracted from one sum and two difference channels. The main difference is that the four signals produced in amplitude comparison monopulse will have similar phases but different amplitudes; however, in phase comparison monopulse the signals have the same amplitude and different phases. Phase comparison monopulse tracking radars use a minimum of a two-element array antenna for each coordinate (azimuth and elevation), as illustrated in Fig. 11.14. A phase error signal (for each coordinate) is computed from the phase difference between the signals generated in the antenna elements.

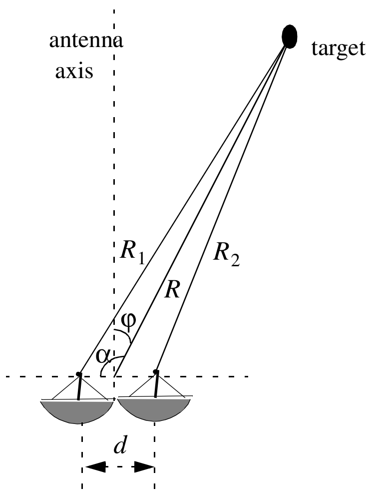


Figure . Single coordinate phase comparison monopulse antenna.

Consider Fig. 11.14; since the angle α is equal to $\varphi + \pi/2$, it follows that

$$R_1^2 = R^2 + \left(\frac{d}{2}\right)^2 - 2\frac{d}{2}R\cos\left(\varphi + \frac{\pi}{2}\right)$$
$$= R^2 + \frac{d^2}{4} - dR\sin\varphi$$

and since $d \ll R$ we can use the binomial series expansion to get

$$R_1 \approx R\left(1 + \frac{d}{2R}\sin\varphi\right)$$

Similarly,

$$R_2 \approx R\left(1 - \frac{d}{2R}\sin\varphi\right)$$

The phase difference between the two elements is then given by

$$\phi = \frac{2\pi}{\lambda}(R_1 - R_2) = \frac{2\pi}{\lambda}d\sin\varphi$$

where λ is the wavelength. The phase difference ϕ is used to determine the angular target location. Note that if $\phi = 0$, then the target would be on the antenna's main axis. The problem with this phase comparison monopulse tech-nique is that it is quite difficult to maintain a stable measurement of the off boresight angle φ, which causes serious performance degradation. This prob-lem can be overcome by implementing a phase comparison monopulse system as illustrated in fig.,

The (single coordinate) sum and difference signals are, respectively, given by

$$\Sigma(\varphi) = S_1 + S_2$$

$$\Delta(\varphi) = S_1 - S_2$$

where the S_1 and S_2 are the signals in the two elements. Now, since S_1 and S_2 have similar amplitude and are different in phase by ϕ, we can write

$$S_1 = S_2 e^{-j\phi}$$

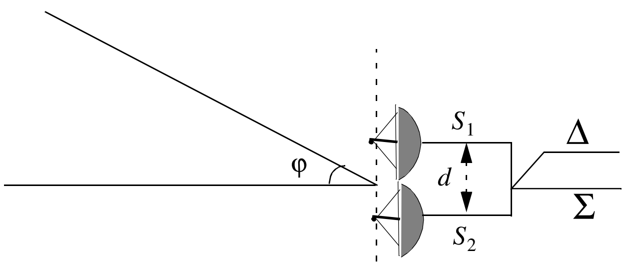


Figure . Single coordinate phase monopulse antenna, with sum and difference channels.

It follows that

$$\Delta(\varphi) = S_2(1 - e^{-j\phi})$$

$$\Sigma(\varphi) = S_2(1 + e^{-j\phi})$$

The phase error signal is computed from the ratio Δ/Σ. More precisely,

$$\frac{\Delta}{\Sigma} = \frac{1 - e^{-j\phi}}{1 + e^{-j\phi}} = j\tan\left(\frac{\phi}{2}\right)$$

which is purely imaginary. The modulus of the error signal is then given by

$$\frac{|\Delta|}{|\Sigma|} = \tan\left(\frac{\phi}{2}\right)$$

This kind of phase comparison monopulse tracker is often called the half-angle tracker.

Range Tracking

Target range is measured by estimating the round-trip delay of the transmit- ted pulses. The process of continuously estimating the range of a moving target is known as range tracking. Since the range to a moving target is changing with time, the range tracker must be constantly adjusted to keep the target locked in range. This can be accomplished using a split gate system, where two range gates (early and late) are utilized. The concept of split gate tracking is illus- trated in fig where a sketch of a typical pulsed radar echo is shown in the figure. The early gate opens at the anticipated starting time of the radar echo and lasts for half its duration. The late gate opens at the center and closes at the end of the echo signal. For this purpose, good estimates of the echo dura- tion and the pulse centertime must be reported to the range tracker so that the early and late gates can be placed properly at the start and center times of the expected echo. This reporting process is widely known as the "designation pro- cess."

The early gate produces positive voltage output while the late gate produces negative voltage output. The outputs of the early and late gates are subtracted, and the difference signal is fed into an integrator to generate an error signal. If both gates are placed properly in time, the integrator output will be equal to zero. Alternatively, when the gates are not timed properly, the integrator output is not zero, which gives an indication that the gates must be moved in time, left or right depending on the sign of the integrator output.

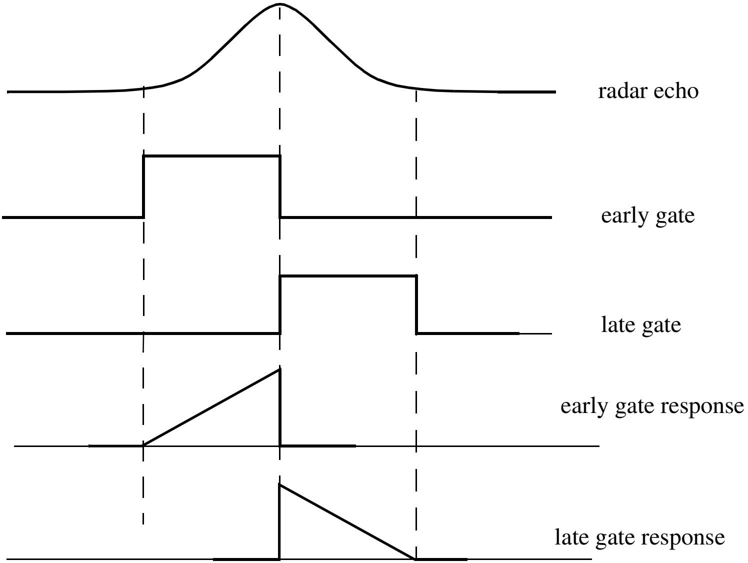


Figure 11.16. Illustration of split-range gate.

Part II: Multiple Target Tracking

Track-while-scan radar systems sample each target once per scan interval, and use sophisticated smoothing and prediction filters to estimate the target parameters between scans. To this end, the Kalman filter and the Alpha-Beta-Gamma ($\alpha\beta\gamma$) filter are commonly used. Once a particular target is detected, the radar may transmit up to a few pulses to verify the target parameters, before it establishes a track file for that target. Target position, velocity, and acceleration comprise the major components of the data maintained by a track file.

The principles of recursive tracking and prediction filters are presented in this part. First, an overview of state representation for Linear Time Invariant (LTI) systems is discussed. Then, second and third order one-dimensional fixed gain polynomial filter trackers are developed. These filters are, respectively, known as the $\alpha\beta$ and $\alpha\beta\gamma$ filters (also known as the g-h and g-h-k filters). Finally, the equations for an n-dimensional multi-state Kalman filter is introduced and analyzed. As a matter of notation, small case letters, with an underneath bar, are used.

Track-While-Scan (TWS)

Modern radar systems are designed to perform multi-function operations, such as detection, tracking, and discrimination. With the aid of sophisticated computer systems, multi-function radars are capable of simultaneously tracking many targets. In this case, each target is sampled once (mainly range and angular position) during a dwell interval (scan). Then, by using smoothing and prediction techniques future samples can be estimated. Radar systems that can perform multi-tasking and multi-target tracking are known as Track-While-Scan (TWS) radars.

Once a TWS radar detects a new target it initiates a separate track file for that detection; this ensures that sequential detections from that target are processed together to estimate the target's future parameters. Position, velocity, and acceleration comprise the main components of the track file. Typically, at least one other confirmation detection (verify detection) is required before the track file is established.

Unlike single target tracking systems, TWS radars must decide whether each detection (observation) belongs to a new target or belongs to a target that has been detected in earlier scans. And in order to accomplish this task, TWS radar systems utilize correlation and association algorithms. In the correlation process each new detection is correlated with all previous detections in order to avoid establishing redundant tracks. If a certain detection correlates with more than one track, then a pre-determined set of association rules are exercised so that the detection is assigned to the proper track. A simplified TWS data pro- cessing block diagram is shown in fig.

Choosing a suitable tracking coordinate system is the first problem a TWS radar has to confront. It is desirable that a fixed reference of an inertial coordinate system be adopted. The radar measurements consist of target range, velocity, azimuth angle, and elevation angle. The TWS system places a gate around the target position and attempts to track the signal within this gate. The gate dimensions are normally azimuth, elevation, and range. Because of the uncertainty associated with the exact target position during the initial detections, a gate has to be large enough so that targets do not move appreciably from scan to scan; more precisely, targets must stay within the gate boundary during successive scans. After the target has been observed for several scans the size of the gate is reduced considerably.

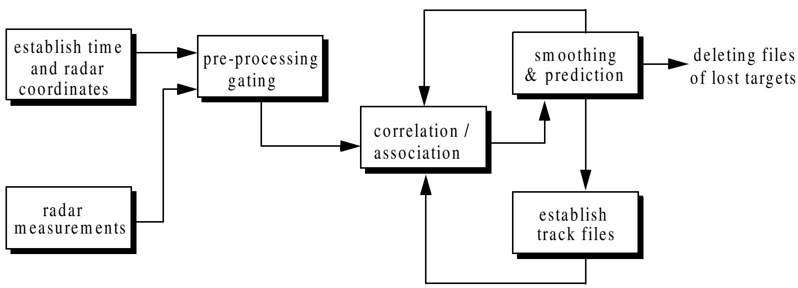


Figure.. Simplified block diagram of TWS data processing.

Gating is used to decide whether an observation is assigned to an existing track file, or to a new track file (new detection). Gating algorithms are normally based on computing a statistical error distance between a measured and an estimated radar observation. For each track file, an upper bound for this error distance is normally set. If the computed difference for a certain radar observation is less than the maximum error distance of a given track file, then the observation is assigned to that track.

All observations that have an error distance less than the maximum distance of a given track are said to correlate with that track. For each observation that does not correlate with any existing tracks, a new track file is established accordingly. Since new detections (measurements) are compared to all existing track files, a track file may then correlate with no observations or with one or more observations. The correlation between observations and all existing track files is identified using a correlation matrix. Rows of the correlation matrix

Substituting Eq. (11.117) into (11.76) and collecting terms the VRR ratios are computed as

$$(VRR)_x = \frac{2\beta(2\alpha^2 + 2\beta - 3\alpha\beta) - \alpha\gamma(4 - 2\alpha - \beta)}{(4 - 2\alpha - \beta)(2\alpha\beta + \alpha\gamma - 2\gamma)} \tag{11.118}$$

$$(VRR)_{\dot{x}} = \frac{4\beta^3 - 4\beta^2\gamma + 2\gamma^2(2 - \alpha)}{T^2(4 - 2\alpha - \beta)(2\alpha\beta + \alpha\gamma - 2\gamma)} \tag{11.119}$$

$$(VRR)_{\ddot{x}} = \frac{4\beta\gamma^2}{T^4(4 - 2\alpha - \beta)(2\alpha\beta + \alpha\gamma - 2\gamma)} \tag{11.120}$$

As in the case of any discrete time system, this filter will be stable if and only if all of its poles fall within the unit circle in the z-plane.

The $\alpha\beta\gamma$ characteristic equation is computed by setting

$$\left|\underline{I} - \underline{A}z^{-1}\right| = 0 \tag{11.121}$$

Substituting Eq. (11.117) into (11.121) and collecting terms yield the following characteristic function:

$$f(z) = z^3 + (-3\alpha + \beta + \gamma)z^2 + (3 - \beta - 2\alpha + \gamma)z - (1 - \alpha) \tag{11.122}$$

The $\alpha\beta\gamma$ becomes a Benedict-Bordner filter when

$$2\beta - \alpha\left(\alpha + \beta + \frac{\gamma}{2}\right) = 0 \tag{11.123}$$

Note that for $\gamma = 0$ Eq. (11.123) reduces to Eq. (11.102). For a critically damped filter the gain coefficients are

$$\alpha = 1 - \xi^3 \tag{11.124}$$

$$\beta = 1.5(1 - \xi^2)(1 - \xi) = 1.5(1 - \xi)^2(1 + \xi) \tag{11.125}$$

$$\gamma = (1 - \xi)^3 \tag{11.126}$$

Note that heavy smoothing takes place when $\xi \rightarrow 1$, while $\xi = 0$ means that no smoothing is present.

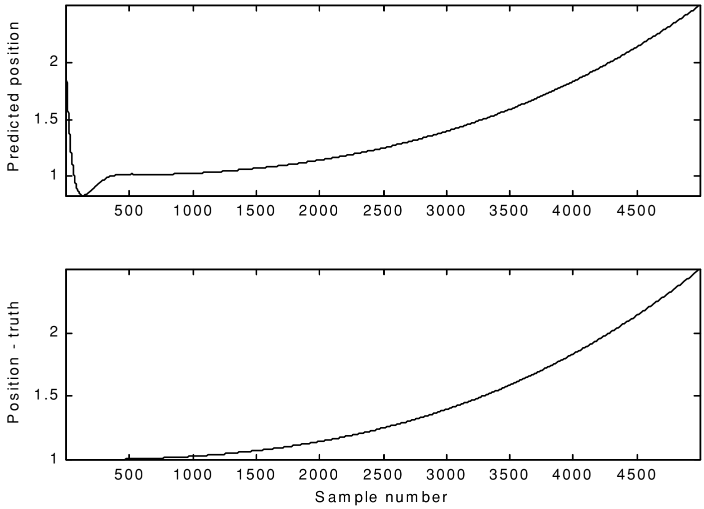


Figure 11.24b-1. Predicted and true position. $\xi = 0.9$ (i.e., small gain coefficients). No noise present.

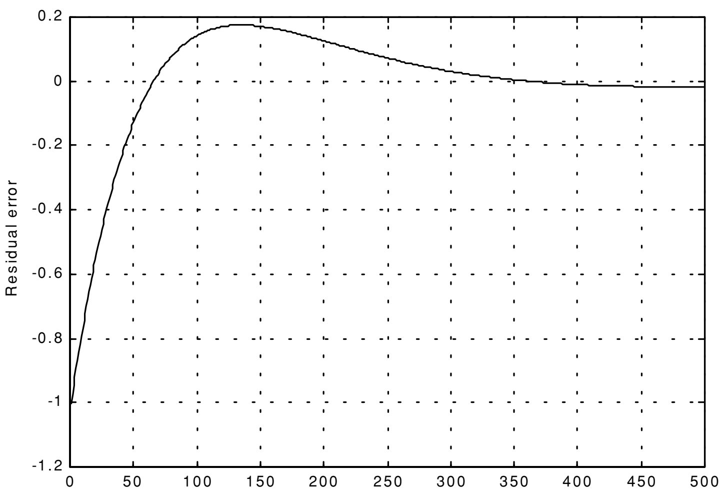


Figure 11.24b-2. Position residual (error). Small gain coefficients. No noise. It takes the filter longer time for the error to settle down.

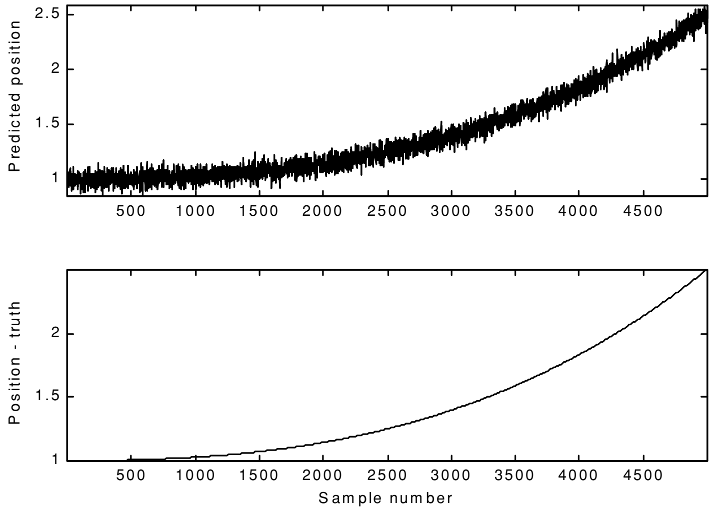


Figure 11.25a-1. Predicted and true position. $\xi = 0.1$ (i.e., large gain coefficients). Noise is present.

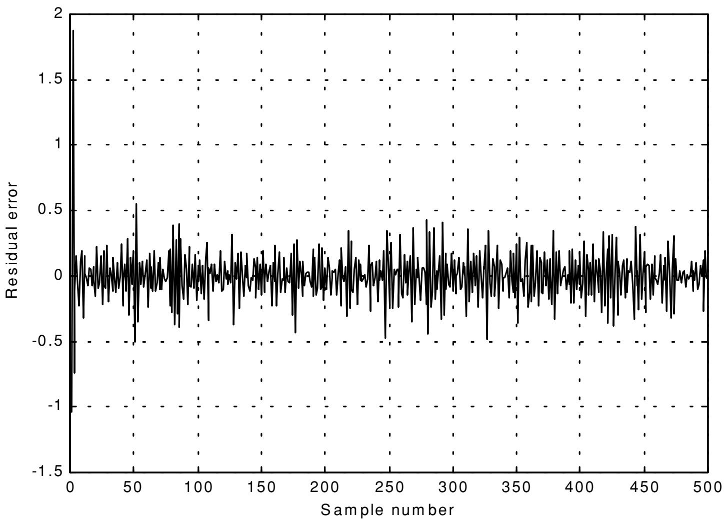


Figure 11.25a-2. Position residual (error). Large gain coefficients. Noise present. The error settles down quickly. The variation is due to noise.

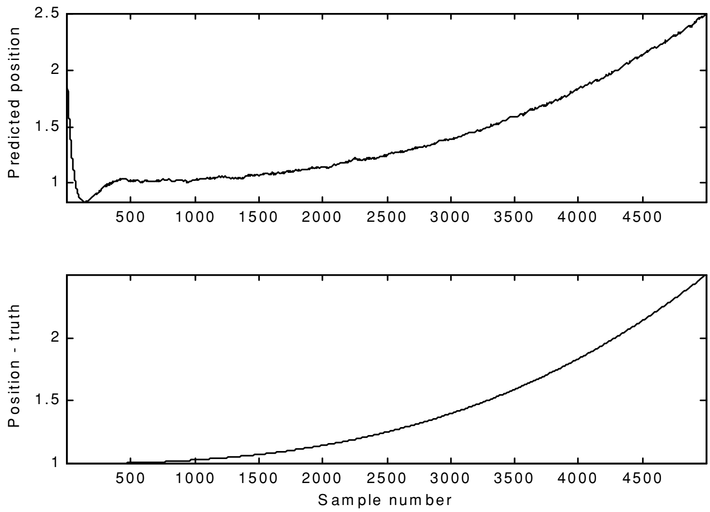


Figure 11.25b-1. Predicted and true position. $\xi = 0.9$ (i.e., small gain coefficients). Noise is present.

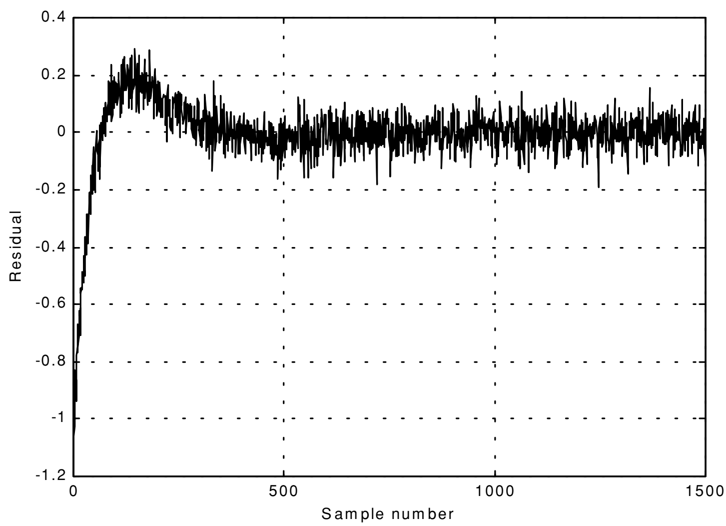


Figure 11.25b-2. Position residual (error). Small gain coefficients. Noise present. The error requires more time before settling down. The variation is due to noise.

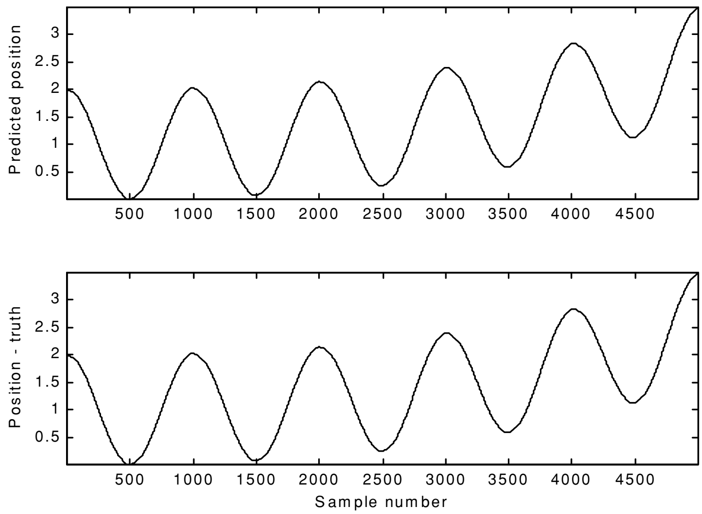


Figure 11.26a. Predicted and true position. $\xi = 0.1$ (i.e., large gain coefficients). Noise is present.

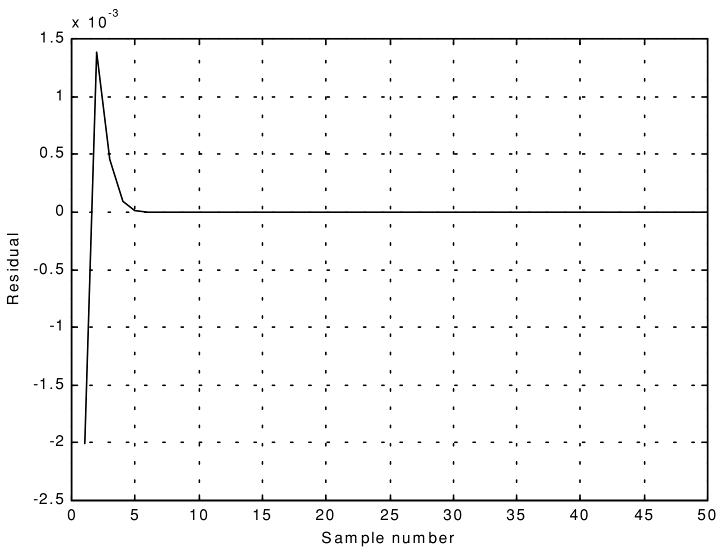


Figure 11.26b. Position residual (error). Large gain coefficients. No noise. The error settles down quickly.

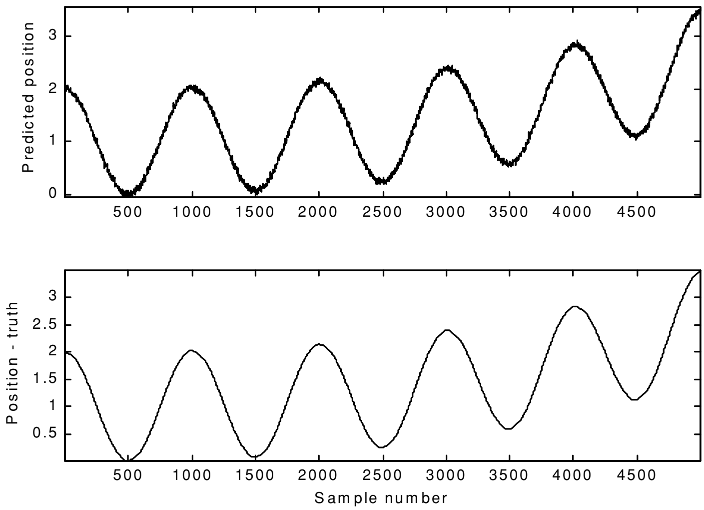


Figure 11.27a. Predicted and true position. $\xi = 0.8$ **(i.e., small gain coefficients). Noise is present.**

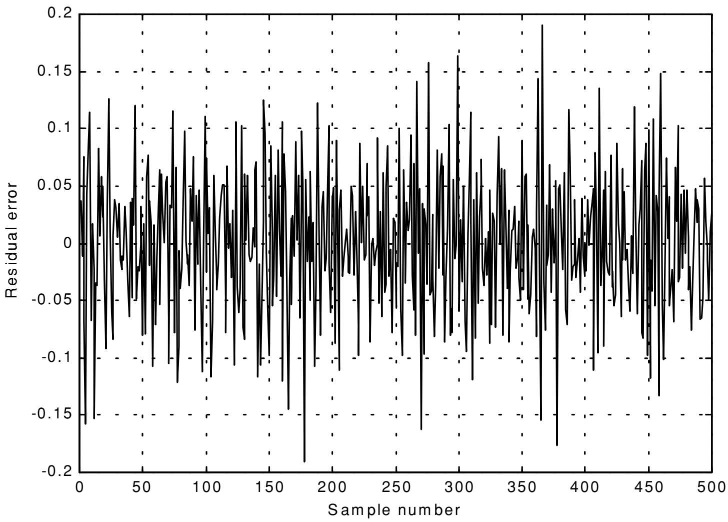


Figure 11.27b. Position residual (error). Small gain coefficients. Noise present. The error stays fairly large; however, its average is around zero. The variation is due to noise.

11.9. The Kalman Filter

The Kalman filter is a linear estimator that minimizes the mean squared error as long as the target dynamics are modeled accurately. All other recursive filters, such as the $\alpha\beta\gamma$ and the Benedict-Bordner filters, are special cases of the general solution provided by the Kalman filter for the mean squared estimation problem. Additionally, the Kalman filter has the following advantages:

1. *The gain coefficients are computed dynamically. This means that the same filter can be used for a variety of maneuvering target environments.*
2. *The Kalman filter gain computation adapts to varying detection histories, including missed detections.*
3. *The Kalman filter provides an accurate measure of the covariance matrix. This allows for better implementation of the gating and association processes.*
4. *The Kalman filter makes it possible to partially compensate for the effects of miss-correlation and miss-association.*

Many derivations of the Kalman filter exist in the literature; only results are provided in this chapter. Fig. 11.28 shows a block diagram for the Kalman filter. The Kalman filter equations can be deduced from Fig. 11.28. The filtering equation is

$$\underline{x}(n|n) = \underline{x}_s(n) = \underline{x}(n|n-1) + K(n)[\underline{y}(n) - \underline{G}\underline{x}(n|n-1)] \tag{11.127}$$

The measurement vector is

$$\underline{y}(n) = \underline{G}\underline{x}(n) + \underline{v}(n) \tag{11.128}$$

where $\underline{v}(n)$ is zero mean, white Gaussian noise with covariance $\Re_c$,

$$\Re_c = E\{\underline{y}(n)\ \underline{y}^t(n)\} \tag{11.129}$$

The gain (weights) vector is dynamically computed as

$$\underline{K}(n) = \underline{P}(n|n-1)\underline{G}^t[\underline{G}\underline{P}(n|n-1)\underline{G}^t + \Re_c]^{-1} \tag{11.130}$$

where the measurement noise matrix $\underline{P}$ represents the predictor covariance matrix, and is equal to

$$\underline{P}(n+1|n) = E\{\underline{x}_s(n+1)\underline{x}^*{}_s(n)\} = \underline{\Phi}\underline{P}(n|n)\underline{\Phi}^t + \underline{Q} \tag{11.131}$$

where $\underline{Q}$ is the covariance matrix for the input $\underline{u}$, where τ_m is the correlation time of the acceleration due to target maneuver or atmospheric turbulence. The correlation time τ_m may vary from as low as 10 seconds for aggressive maneuvering to as large as 60 seconds for lazy maneuver cases.

Singer defined the random target acceleration model by a first order Markov process given by

$$\ddot{x}(n+1) = \rho_m \ \ddot{x}(n) + \sqrt{1-\rho_m^2} \ \sigma_m \ w(n) \tag{11.136}$$

where $w(n)$ is a zero mean, Gaussian random variable with unity variance, σ_m is the maneuver standard deviation, and the maneuvering correlation coefficient ρ_m is given by

$$\rho_m = e^{-\frac{T}{\tau_m}} \tag{11.137}$$

The continuous time domain system that corresponds to these conditions is as the Wiener-Kolmogorov whitening filter which is defined by the differential equation

$$\frac{d}{dt}v(t) = -\beta_m v(t) + w(t) \tag{11.138}$$

where β_m is equal to $1/\tau_m$. The maneuvering variance using Singer's model is given by

$$\sigma_m^2 = \frac{A_{max}^2}{3}[1 + 4P_{max} - P_0] \tag{11.139}$$

A_{max} is the maximum target acceleration with probability P_{max} and the term P_0 defines the probability that the target has no acceleration.

The transition matrix that corresponds to the Singer filter is given by

$$\underline{\Phi} = \begin{bmatrix} 1 & T & \frac{1}{\beta_m^2}(-1 + \beta_m T + \rho_m) \\ 0 & 1 & \frac{1}{\beta_m}(1-\rho_m) \\ 0 & 0 & \rho_m \end{bmatrix} \tag{11.140}$$

Note that when $T\beta_m = T/\tau_m$ is small (the target has constant acceleration), then Eq. (11.140) reduces to Eq. (11.114). Typically, the sampling interval T is much less than the maneuver time constant τ_m; hence, Eq. (11.140) can be accurately replaced by its second order approximation. More precisely,

$$\underline{\Phi} = \begin{bmatrix} 1 & T & T^2/2 \\ 0 & 1 & T(1 - T/2\tau_m) \\ 0 & 0 & \rho_m \end{bmatrix} \tag{11.141}$$

The covariance matrix was derived by Singer, and it is equal to

$$\underline{C} = \frac{2\sigma_m^2}{\tau_m} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \tag{11.142}$$

where

$$C_{11} = \sigma_x^2 = \frac{1}{2\beta_m^5}\left[1 - e^{-2\beta_m T} + 2\beta_m T + \frac{2\beta_m^3 T^3}{3} - 2\beta_m^2 T^2 - 4\beta_m T e^{-\beta_m T}\right] \tag{11.143}$$

$$C_{12} = C_{21} = \frac{1}{2\beta_m^4}[e^{-2\beta_m T} + 1 - 2e^{-\beta_m T} + 2\beta_m T e^{-\beta_m T} - 2\beta_m T + \beta_m^2 T^2] \tag{11.144}$$

$$C_{13} = C_{31} = \frac{1}{2\beta_m^3}[1 - e^{-2\beta_m T} - 2\beta_m T e^{-\beta_m T}] \tag{11.145}$$

$$C_{22} = \frac{1}{2\beta_m^3}[4e^{-\beta_m T} - 3 - e^{-2\beta_m T} + 2\beta_m T] \tag{11.146}$$

$$C_{23} = C_{32} = \frac{1}{2\beta_m^2}[e^{-2\beta_m T} + 1 - 2e^{-\beta_m T}] \tag{11.147}$$

$$C_{33} = \frac{1}{2\beta_m}[1 - e^{-2\beta_m T}] \tag{11.148}$$

Two limiting cases are of interest:

1. *The short sampling interval case ($T \ll \tau_m$),*

$$\lim_{\beta_m T \to 0} \underline{C} = \frac{2\sigma_m^2}{\tau_m} \begin{bmatrix} T^5/20 & T^4/8 & T^3/6 \\ T^4/8 & T^3/3 & T^2/2 \\ T^3/6 & T^2/2 & T \end{bmatrix} \tag{11.149}$$

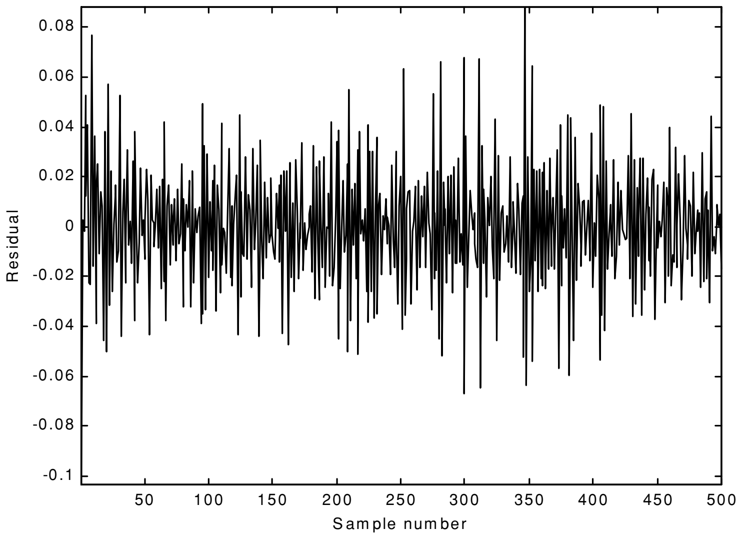


Figure 11.29b. Residual corresponding to Fig. 11.29a.

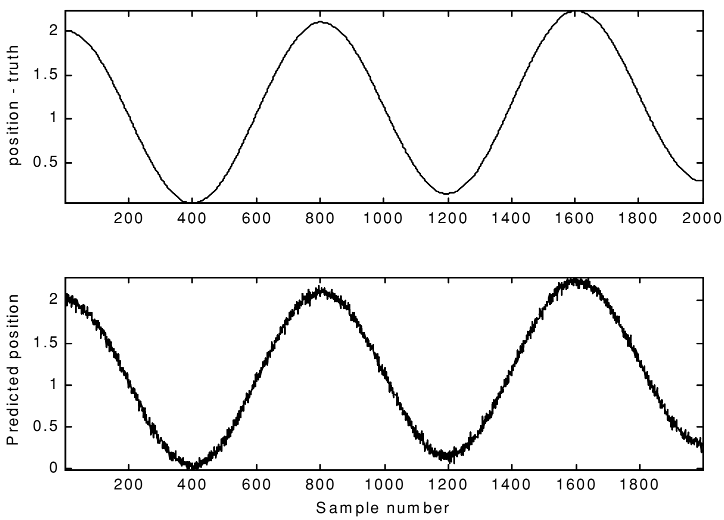


Figure 11.30a. True and predicted positions. Aggressive maneuvering. Plot produced using the function *"kalman_filter.m"*.

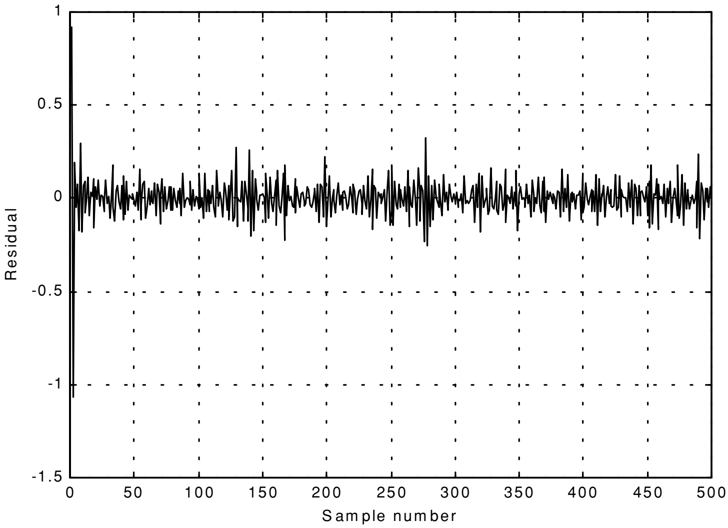


Figure 11.30b. Residual corresponding to Fig. 11.30a.

11.10. MATLAB Programs and Functions

This section contains listings of all MATLAB programs and functions used in this chapter. Users are encouraged to rerun these codes with different inputs in order to enhance their understanding of the theory.

Listing 11.1. MATLAB Function "mono_pulse.m"

```
function mono_pulse(phi0)
eps = 0.0000001;
angle = -pi:0.01:pi;
y1 = sinc(angle + phi0);
y2 = sinc((angle - phi0));
ysum = y1 + y2;
ydif = -y1 + y2;
figure (1)
plot (angle,y1,'k',angle,y2,'k');
grid;
xlabel ('Angle - radians')
ylabel ('Squinted patterns')
figure (2)
plot(angle,ysum,'k');
```

```
xlabel ('Sample number')
ylabel ('Predicted position')
```

Problems

11.1. Show that in order to be able to quickly achieve changing the beam position the error signal needs to be a linear function of the deviation angle.

11.2. Prepare a short report on the vulnerability of conical scan to amplitude modulation jamming. In particular consider the self-protecting technique called "Gain Inversion."

11.3. Consider a conical scan radar. The pulse repetition interval is $10\mu s$. Calculate the scan rate so that at least ten pulses are emitted within one scan.

11.4. Consider a conical scan antenna whose rotation around the tracking axis is completed in 4 seconds. If during this time 20 pulses are emitted and received, calculate the radar PRF and the unambiguous range.

11.5. Reproduce Fig. 11.11 for $\varphi_0 = 0.05, 0.1, 0.15$ radians.

11.6. Reproduce Fig. 11.13 for the squint angles defined in the previous problem.

11.7. Derive Eq. (11.33) and Eq. (11.34).

11.8. Consider a monopulse radar where the input signal is comprised of both target return and additive white Gaussian noise. Develop an expression for the complex ratio Σ/Δ.

11.9. Consider the sum and difference signals defined in Eqs. (11.7) and (11.8). What is the squint angle φ_0 that maximizes $\Sigma(\varphi = 0)$?

11.10. A certain system is defined by the following difference equation:

$$y(n) + 4y(n-1) + 2y(n-2) = w(n)$$

Find the solution to this system for $n > 0$ and $w = \delta$.

11.11. Prove the state transition matrix properties (i.e., Eqs. (11.30) through (11.36)).

11.12. Suppose that the state equations for a certain discrete time LTI system are

$$\begin{bmatrix} x_1(n+1) \\ x_2(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(n)$$

If $y(0) = y(1) = 1$, find $y(n)$ when the input is a step function.

11.13. Derive Eq. (11.55).

11.14. Derive Eq. (11.75).