UNIT- IV

TERMINOLOGY

Construction of Shape Function by Degrading Technique	The geometry of the structure or its loading and boundary conditions are such that the stresses developed in few locations are quite high. On the other hand, variations of stresses are less in some areas and as a result, refinement of finite element mesh is not necessary. It would be economical in terms of computation if higher order elements are chosen where stress concentration is high and lower order elements at area away from the critical area.
Element Properties of Rectangular Elements	Rectangular elements are suitable for modeling regular geometries. Sometimes, it is used along with triangular elements to represent an arbitrary geometry. The simplest element in the rectangular family is the four node rectangle with sides parallel to x and y axis.
Shape Function for Four Node Element	Shape functions of a rectangular element can be derived using both Cartesian and natural coordinate systems. A four term polynomial expression for the field variable will be required for a rectangular element with four nodes having four degrees of freedom. Since there is no complete four term polynomial in two dimensions, the incomplete, symmetric expression from the Pascal's triangle may be chosen to ensure geometric isotropy.
Shape function using natural coordinates	The derivation of interpolation function in terms of Cartesian coordinate system is algebraically complex as seen from earlier section. However, the complexity can be reduced by the use of natural coordinate system, where the natural coordinates will vary from -1 to +1 in place of –a to +a or –b to +b.
Shape Function for Eight Node Element	The shape function of eight node rectangular element can be derived in similar fashion as done in case of four node element. The only difference will be on choosing of polynomial as this element is of quadratic in nature. The derivation will be algebraically complex in case of using Cartesian coordinate system. However, use of the natural coordinate system will

	make the process simpler as the natural coordinates vary from -1 to +1 in the element.
Serendipity Elements	Higher order Lagrange elements contains internal nodes, which do not contribute to the inter element connectivity. However, these can be eliminated by condensation procedure which needs extra computation. The elimination of these internal nodes results in reduction in size of the element matrices. Alternatively, one can develop shape functions of two dimensional elements which contain nodes only on the boundaries.
Tetrahedral Elements	The simplest element of the tetrahedral family is a four node tetrahedron. The node numbering has been followed in sequential manner, i.e, in this case anti-clockwise direction. Similar to the area coordinates, the concept of volume coordinates has been introduced here. The coordinates of the nodes are defined both in Cartesian and volume coordinates. Point P(x,y, and z) is an arbitrary point in the tetrahedron.
Isoparametric element	As the same shape functions are used for both the field variable and description of element geometry, the method is known as isoparametric mapping. The element defined by such a method is known as an isoparametric element. This method can be used to transform the natural coordinates of a point to the Cartesian coordinate system and vice versa.

Concepts

Evaluation of Stiffness Matrix of 2-D Isoparametric Elements

For two dimensional plane stress/strain formulation, the strain vector can be represented as

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \end{Bmatrix} = \begin{Bmatrix} J_{11}^* \cdot \frac{\partial u}{\partial \xi} + J_{12}^* \cdot \frac{\partial u}{\partial \eta} \\ J_{21}^* \cdot \frac{\partial v}{\partial \xi} + J_{22}^* \cdot \frac{\partial v}{\partial \eta} \\ J_{11}^* \cdot \frac{\partial v}{\partial \xi} + J_{12}^* \cdot \frac{\partial v}{\partial \eta} + J_{21}^* \cdot \frac{\partial u}{\partial \xi} + J_{22}^* \cdot \frac{\partial u}{\partial \eta} \end{Bmatrix}$$

The above expression can be rewritten in matrix form

$$\{\varepsilon\} = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix}$$

For an n node element the displacement u can be represented as,

$$u = \sum_{i=1}^{n} N_i u_i$$

and similarly for v & w.

Thus,

$$
\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial N_1}{\partial \xi} & \cdots & \dfrac{\partial N_n}{\partial \xi} & 0 & \cdots & 0 \\ \dfrac{\partial N_1}{\partial \eta} & \cdots & \dfrac{\partial N_n}{\partial \eta} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \dfrac{\partial N_1}{\partial \xi} & \cdots & \dfrac{\partial N_n}{\partial \xi} \\ 0 & \cdots & 0 & \dfrac{\partial N_1}{\partial \eta} & \cdots & \dfrac{\partial N_n}{\partial \eta} \end{bmatrix} \begin{Bmatrix} u_1 \\ \vdots \\ u_n \\ v_1 \\ \vdots \\ v_n \end{Bmatrix}
$$

As a result, above equations can be written which will be as follows.

$$
\{\varepsilon\} = \begin{bmatrix} J^*_{11} & J^*_{12} & 0 & 0 \\ 0 & 0 & J^*_{21} & J^*_{22} \\ J^*_{21} & J^*_{22} & J^*_{11} & J^*_{12} \end{bmatrix} \begin{bmatrix} \dfrac{\partial N_1}{\partial \xi} & \cdots & \dfrac{\partial N_n}{\partial \xi} & 0 & \cdots & 0 \\ \dfrac{\partial N_1}{\partial \eta} & \cdots & \dfrac{\partial N_n}{\partial \eta} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \dfrac{\partial N_1}{\partial \xi} & \cdots & \dfrac{\partial N_n}{\partial \xi} \\ 0 & \cdots & 0 & \dfrac{\partial N_1}{\partial \eta} & \cdots & \dfrac{\partial N_n}{\partial \eta} \end{bmatrix} \begin{Bmatrix} u_1 \\ : \\ : \\ u_n \\ v_1 \\ : \\ : \\ v_n \end{Bmatrix}
$$

Or,

$$
\{\varepsilon\} = [B]\{d\}
$$

Where {d} is the nodal displacement vector and [B] is known as strain displacement relationship matrix and can be obtained as

$$[B]=\begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* \end{bmatrix}\begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \cdots & \frac{\partial N_n}{\partial \xi} & 0 & \cdots & 0 \\ \frac{\partial N_1}{\partial \eta} & \cdots & \frac{\partial N_n}{\partial \eta} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{\partial N_1}{\partial \xi} & \cdots & \frac{\partial N_n}{\partial \xi} \\ 0 & \cdots & 0 & \frac{\partial N_1}{\partial \eta} & \cdots & \frac{\partial N_n}{\partial \eta} \end{bmatrix}$$

It is necessary to transform integrals from Cartesian to the natural coordinates as well for calculation of the elemental stiffness matrix in isoparametric formulation. The differential area relationship can be established from advanced calculus and the elemental area in Cartesian coordinate can be represented in terms of area in natural coordinates as:

dA = dx dy = [J] dx dh

Here J is the determinant of the Jacobian matrix. The stiffness matrix for a two dimensional element may be expressed as

$$[k]=\iiint_{\Omega}[B]^T[D][B]d\Omega = t\iint_{A}[B]^T[D][B]dxdy$$

Here, [B] is the strain-displacement relationship matrix and t is the thickness of the element. The above expression in Cartesian coordinate system can be changed to the natural coordinate system as follows to obtain the elemental stiffness matrix

$$[k]=t\int_{-1}^{+1}\int_{-1}^{+1}[B]^T[D][B]|J|d\xi d\eta$$

Though the isoparametric formulation is mathematically straightforward, the algebraic difficulty is significant.

Evaluation of Stiffness Matrix of 3-D Isoparametric Elements

Stiffness matrix of 3-D solid isoparametric elements can easily be formulated by the extension of the procedure followed for plane elements. For example, the eight node solid element is analogous to the four node plane element. The strain vector for solid element can be written in the following form.

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial z} \end{Bmatrix}$$

The above equation can be expressed as

$$\{\varepsilon\} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y} \\ \dfrac{\partial x}{\partial z} + \dfrac{\partial w}{\partial x} \end{Bmatrix} = \begin{bmatrix} J_{11}^* & J_{12}^* & J_{13}^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & J_{21}^* & J_{22}^* & J_{23}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & J_{31}^* & J_{32}^* & J_{33}^* \\ J_{21}^* & J_{22}^* & J_{23}^* & J_{11}^* & J_{12}^* & J_{13}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & J_{31}^* & J_{32}^* & J_{33}^* & J_{21}^* & J_{22}^* & J_{23}^* \\ J_{31}^* & J_{32}^* & J_{33}^* & 0 & 0 & 0 & J_{11}^* & J_{12}^* & J_{13}^* \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial u}{\partial \zeta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \\ \dfrac{\partial v}{\partial \zeta} \\ \dfrac{\partial w}{\partial \xi} \\ \dfrac{\partial w}{\partial \eta} \\ \dfrac{\partial v}{\partial \zeta} \end{Bmatrix}$$

For an 8 node brick element u can be represented as, $u = \sum_{i=1}^{8} N_i u_i$ and similarly for v&w.

$$\frac{\partial u}{\partial \xi} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi} u_i \, , \; \frac{\partial u}{\partial \eta} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \eta} u_i \;\&\; \frac{\partial u}{\partial \zeta} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \zeta} u_i$$

$$\frac{\partial v}{\partial \xi} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi} v_i \, , \; \frac{\partial v}{\partial \eta} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \eta} v_i \;\&\; \frac{\partial v}{\partial \zeta} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \zeta} v_i$$

$$\frac{\partial w}{\partial \xi} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi} w_i \, , \; \frac{\partial w}{\partial \eta} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \eta} w_i \;\&\; \frac{\partial w}{\partial \zeta} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \zeta} w_i$$

Hence eq. can be rewritten as

$$\{\varepsilon\} = \begin{bmatrix} J^*_{11} & J^*_{12} & J^*_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & J^*_{21} & J^*_{22} & J^*_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & J^*_{31} & J^*_{32} & J^*_{33} \\ J^*_{21} & J^*_{22} & J^*_{23} & J^*_{11} & J^*_{12} & J^*_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & J^*_{31} & J^*_{32} & J^*_{33} & J^*_{21} & J^*_{22} & J^*_{23} \\ J^*_{31} & J^*_{32} & J^*_{33} & 0 & 0 & 0 & J^*_{11} & J^*_{12} & J^*_{13} \end{bmatrix} \times \sum_{i=1}^{8} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial \eta} & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial \zeta} \\ \frac{\partial N_i}{\partial \eta} & \frac{\partial N_i}{\partial \xi} & 0 \\ 0 & \frac{\partial N_i}{\partial \zeta} & \frac{\partial N_i}{\partial \eta} \\ \frac{\partial N_i}{\partial \zeta} & 0 & \frac{\partial N_i}{\partial \xi} \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix}$$

Thu, the strain-displacement relationship matrix [B] for 8 node brick element is

Thus, the strain-displacement relationship matrix [B] for 8 node brick element is

$$[B] = \begin{bmatrix} J^*_{11} & J^*_{12} & J^*_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & J^*_{21} & J^*_{22} & J^*_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & J^*_{31} & J^*_{32} & J^*_{33} \\ J^*_{21} & J^*_{22} & J^*_{23} & J^*_{11} & J^*_{12} & J^*_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & J^*_{31} & J^*_{32} & J^*_{33} & J^*_{21} & J^*_{22} & J^*_{23} \\ J^*_{31} & J^*_{32} & J^*_{33} & 0 & 0 & 0 & J^*_{11} & J^*_{12} & J^*_{13} \end{bmatrix} \times \sum_{i=1}^{8} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial \eta} & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial \zeta} \\ \frac{\partial N_i}{\partial \eta} & \frac{\partial N_i}{\partial \xi} & 0 \\ 0 & \frac{\partial N_i}{\partial \zeta} & \frac{\partial N_i}{\partial \eta} \\ \frac{\partial N_i}{\partial \zeta} & 0 & \frac{\partial N_i}{\partial \xi} \end{bmatrix}$$

The stiffness matrix may be found by using the following expression in natural coordinate system.

$$[\mathrm{k}] = \iiint_{\Omega} [\mathrm{B}]^{\mathrm{T}} [\mathrm{D}][\mathrm{B}] \mathrm{d}\Omega = \iiint_{V} [\mathrm{B}]^{\mathrm{T}} [\mathrm{D}][\mathrm{B}] \mathrm{dxdydz} = \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} [\mathrm{B}]^{\mathrm{T}} [\mathrm{D}][\mathrm{B}] \mathrm{d}\xi \mathrm{d}\eta \mathrm{d}\zeta |\mathrm{J}|$$

Important Questions

1. State Concepts of Isoparametric Formulation.
2. Explain about isoparametric elements for 2D analysis.
3. Derive formulation of CST element.
4. Derive 4 –Noded iso-parametric elements.
5. Derive 8-noded iso-parametric quadrilateral elements.
6. Write about Lagrangian and Serendipity elements.
7. Explain Basic principles of Axi-Symmetric Analysis.
8. Derive Formulation of 4-noded iso-parametric axi-symmetric element.