UNIT- V

TERMINOLOGY

Jacobian Matrix	A variety of derivatives of the interpolation functions with respect to the global coordinates are necessary to formulate the element stiffness matrices. As the both element geometry and variation of the shape functions are represented in terms of the natural coordinates of the parent element, some additional mathematical obstacle arises. For example, in case of evaluation of the strain vector, the operator matrix is with respect to x and y, but the interpolation function is with x and h. Therefore, the operator matrix is to be transformed for taking derivative with x and h.
Numerical Integration for One Dimensional	The integrations, we generally encounter in finite element methods, are quite complicated and it is not possible to find a closed form solutions to those problems. Exact and explicit evaluation of the integral associated to the element matrices and the loading vector is not always possible because of the algebraic complexity of the coefficient of the different equation (i.e., the stiffness influence coefficients, elasticity matrix, loading functions etc.).
Generalized Stiffness Matrix	The generalized stiffness matrix of a grid member can be obtained by transferring the matrix of local coordinate system into its global coordinate system.
Constant Strain Triangle	The triangular elements with different numbers of nodes are used for solving two dimensional solid members. The linear triangular element was the first type of element developed for the finite element analysis of 2D solids. However, it is observed that the linear triangular element is less accurate compared to linear quadrilateral elements. But the triangular element is still a very useful element for its adaptivity to complex geometry. These are used if the geometry of the 2D model is complex in nature. Constant strain triangle (CST) is the simplest element to develop mathematically.

Element Stiffness Matrix for LST	The elements are able to provide enough information about displacement pattern of the element, but it is unable to provide adequate information about stress inside an element. This limitation will be significant enough in regions of high strain gradients. The use of a higher order triangular element called Linear Strain Triangle (LST) significantly improves the results at these areas as the strain inside the element is varying.
Axisymmetric Element	If the problem geometry is symmetric about an axis and the loading and boundary conditions are symmetric about the same axis, the problem is said to be axisymmetric. Such three-dimensional problems can be solved using two-dimensional finite elements.
Element Load Vector	The forces on an element can be generated due to its self -weight or externally applied force which may be concentrated or distributed in nature. The distributed load may be uniform or non-uniform. All these types of loads are redistributed to the nodes using finite element formulation.
Shell	A shell is a curved surface, which by virtue of their shape can withstand both membrane and bending forces. A shell structure can take higher loads if, membrane stresses are predominant, which is primarily caused due to in-plane forces (plane stress condition).

Concepts

Lagrange Interpolation Function

An alternate and simpler way to derive shape functions is to use Lagrange interpolation polynomials. This method is suitable to derive shape function for elements having higher order of nodes. The Lagrange interpolation function at node I is defined by

$$f_i(\xi) = \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{(\xi - \xi_j)}{(\xi_i - \xi_j)} = \frac{(\xi - \xi_1)(\xi - \xi_2)\ldots(\xi - \xi_{i-1})(\xi - \xi_{i+1})\ldots(\xi - \xi_n)}{(\xi_i - \xi_1)(\xi_i - \xi_2)\ldots(\xi_i - \xi_{i-1})(\xi_i - \xi_{i+1})\ldots(\xi_i - \xi_n)}$$

The function fi(ξ) produces the Lagrange interpolation function for i th node, and ξj denotes ξ coordinate of j th node in the element. In the above equation if we put ξ = ξj, and j ≠ i, the value of the function fi(ξ) will be equal to zero. Similarly, putting ξ = ξi, the numerator will be equal to denominator and hence fi(ξ) will have a value of unity. Since, Lagrange interpolation function for i th node includes product of all terms except j th term; for an element with n nodes, fi(ξ) will have n-1 degrees of freedom. Thus, for one-dimensional elements with n-nodes we can define shape function as i i N (x)=f (x).

Serendipity Elements

Higher order Lagrange elements contains internal nodes, which do not contribute to the inter element connectivity. However, these can be eliminated by condensation procedure which needs extra computation. The elimination of these internal nodes results in reduction in size of the element matrices. Alternatively, one can develop shape functions of two dimensional elements which contain nodes only on the boundaries. These elements are called serendipity elements.

The interpolation function can be derived by inspection in terms of natural coordinate system as follows:

(a) Linear element

$$N_i(\xi, \eta) = \frac{1}{4}(1 + \xi\xi_i)(1 + \eta\eta_i)$$

(b) Quadratic element

(i) For nodes at $\xi = \pm 1,\ \eta = \pm 1$

$$N_i(\xi,\eta) = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)(\xi\xi_i+\eta\eta_i-1)$$

(ii) For nodes at $\xi = \pm 1,\ \eta = 0$

$$N_i(\xi,\eta) = \frac{1}{2}(1+\xi\xi_i)(1-\eta^2)$$

(iii) For nodes at $\xi = 0,\ \eta = \pm 1$

$$N_i(\xi,\eta) = \frac{1}{2}(1-\xi^2)(1+\eta\eta_i)$$

(c) Cubic element

(i) For nodes at $\xi = \pm 1,\ \eta = \pm 1$

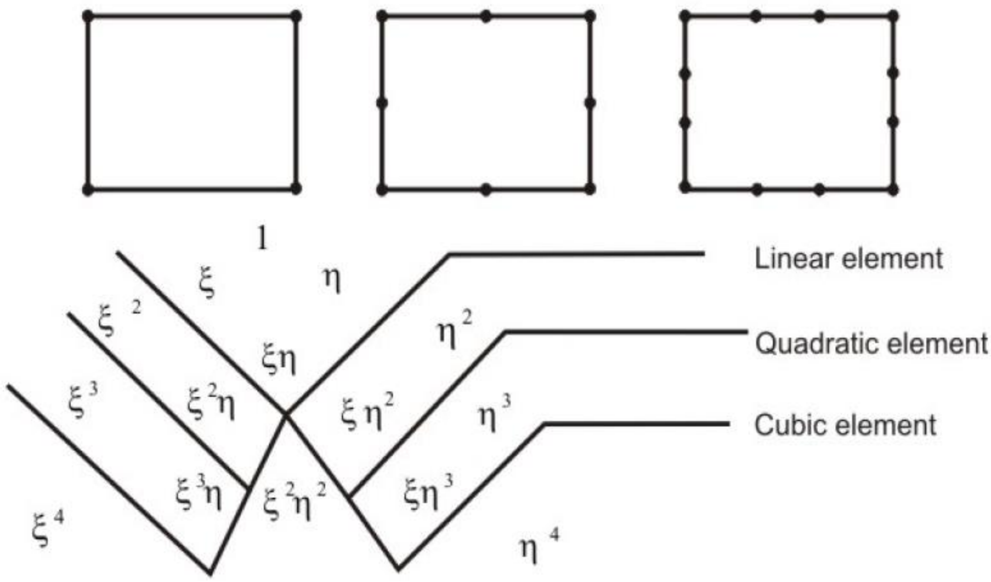
$$N_i(\xi,\eta) = \frac{1}{32}(1+\xi\xi_i)(1+\eta\eta_i)\left[9(\xi^2+\eta^2)-10\right]$$

(ii) For nodes at $\xi = \pm 1,\ \eta = \pm \frac{1}{3}$

$$N_i(\xi,\eta) = \frac{9}{32}(1+\xi\xi_i)(1-\eta^2)(1+9\eta\eta_i)$$

And so on for other nodes at the boundaries.

Thus, the nodal conditions must be satisfied by each interpolation function to obtain the functions serendipitously.



Two dimensional serendipity elements and Pascal triangle

Numerical Integration

The integrations, we generally encounter in finite element methods, are quite complicated and it is not possible to find a closed form solutions to those problems. Exact and explicit evaluation of the integral associated to the element matrices and the loading vector is not always possible because of the algebraic complexity of the coefficient of the different equation (i.e., the stiffness influence coefficients, elasticity matrix, loading functions etc.). In the finite element analysis, we face the problem of evaluating the following types of integrations in one, two and three dimensional cases respectively. These are necessary to compute element stiffness and element load vector.

$$\int \phi(\xi)\,d\xi; \quad \int \phi(\xi,\eta)\,d\xi d\eta; \quad \int \phi(\xi,\eta,\zeta)\,d\xi d\eta d\zeta$$

Approximate solutions to such problems are possible using certain numerical techniques. Several numerical techniques are available, in mathematics for solving definite integration problems, including, mid-point rule, trapezoidal-rule, Simpson's 1/3rd rule, Simpson's 3/8th rule and Gauss Quadrature formula. Among these, Gauss Quadrature technique is most useful one for solving problems in finite element method.

Gauss Quadrature for One-Dimensional Integrals

The concept of Gauss Quadrature is first illustrated in one dimension in the context of an integral in the form of

$$I = \int_{-1}^{+1} \phi(\xi)d\xi \quad \text{from} \quad \int_{x_1}^{x_2} f(x)dx.$$

To transform from an arbitrary interval of $x1 \leq x \leq x2$ to an interval of $-1 \leq \xi \leq 1$, we need to change the integration function from f(x) to $\phi(\xi)$ accordingly. Thus, for a linear variation in one dimension, one can write the following relations.

$$x = \frac{1-\xi}{2}x_1 + \frac{1+\xi}{2}x_2 = N_1x_1 + N_2x_2$$

$$\text{so for } \xi = -1, x = \frac{1-(-1)}{2}x_1 + \frac{1-1}{2}x_2 = x_1$$

$$\xi = +1, \quad x = x_2$$

$$\therefore I = \int_{x_1}^{x_2} f(x)dx = \int_{-1}^{+1} \phi(\xi)d\xi$$

Numerical integration based on Gauss Quadrature assumes that the function $\varphi(\xi)$ will be evaluated over an interval $-1 \leq \xi \leq 1$. Considering an one-dimensional integral, Gauss Quadrature represents the integral $\varphi(\xi)$ in the form of

$$I = \int_{-1}^{+1} \phi(\xi)d\xi \approx \sum_{i=1}^{n} w_i\phi(\xi_i) \approx w_1\phi(\xi_1) + w_2\phi(\xi_2) + \ldots + w_\eta\phi(\xi_\eta)$$

Where, the $\xi 1, \xi 2, \xi 3, \ldots, \xi n$ represents n numbers of points known as Gauss Points and the corresponding coefficients w1, w2, w3, …, wn are known as weights. The location and weight coefficients of Gauss points are calculated by Legendre polynomials. Hence this method is also sometimes referred as Gauss-Legendre Quadrature method. The summation of these values at n sampling points gives the exact solution of a polynomial integrand of an order up to 2n-1. For example, considering sampling at two Gauss points we can get exact solution for a polynomial of an order (2×2-1) or 3. The use of more number of Gauss points has no effect on accuracy of results but takes more computation time.

One- Point Formula

Considering n = 1, then

$$\int_{-1}^{1} \phi(\xi)d\xi \approx w_1\phi(\xi_1)$$

Since there are two parameters W1 and ξ1, we need a first order polynomial for φ(ξ) to evaluate the equation exactly. For example, considering,

$$\phi(\xi) = a_0 + a_1\xi,$$

$$\text{Error} = \int_{-1}^{1} (a_0 + a_1\xi)d\xi - w_1\phi(\xi_1) = 0$$

$$\Rightarrow 2a_0 - w_1(a_0 + a_1\xi_1) = 0$$

$$\Rightarrow a_0(2 - w_1) - w_1a_1\xi_1 = 0$$

Thus, the error will be zero if W1 =2 and ξ1=0. Putting these in eq.(3.8.3), for any general φ, we have

$$I = \int_{-1}^{1} \phi(\xi)d\xi = 2\phi(0)$$

This is exactly similar to the well-known midpoint rule.

Two-Point Formula

If we consider n = 2, then

$$\int_{-1}^{1} \phi(\xi)\, d\xi \approx w_1 \phi(\xi_1) + w_2 \phi(\xi_2)$$

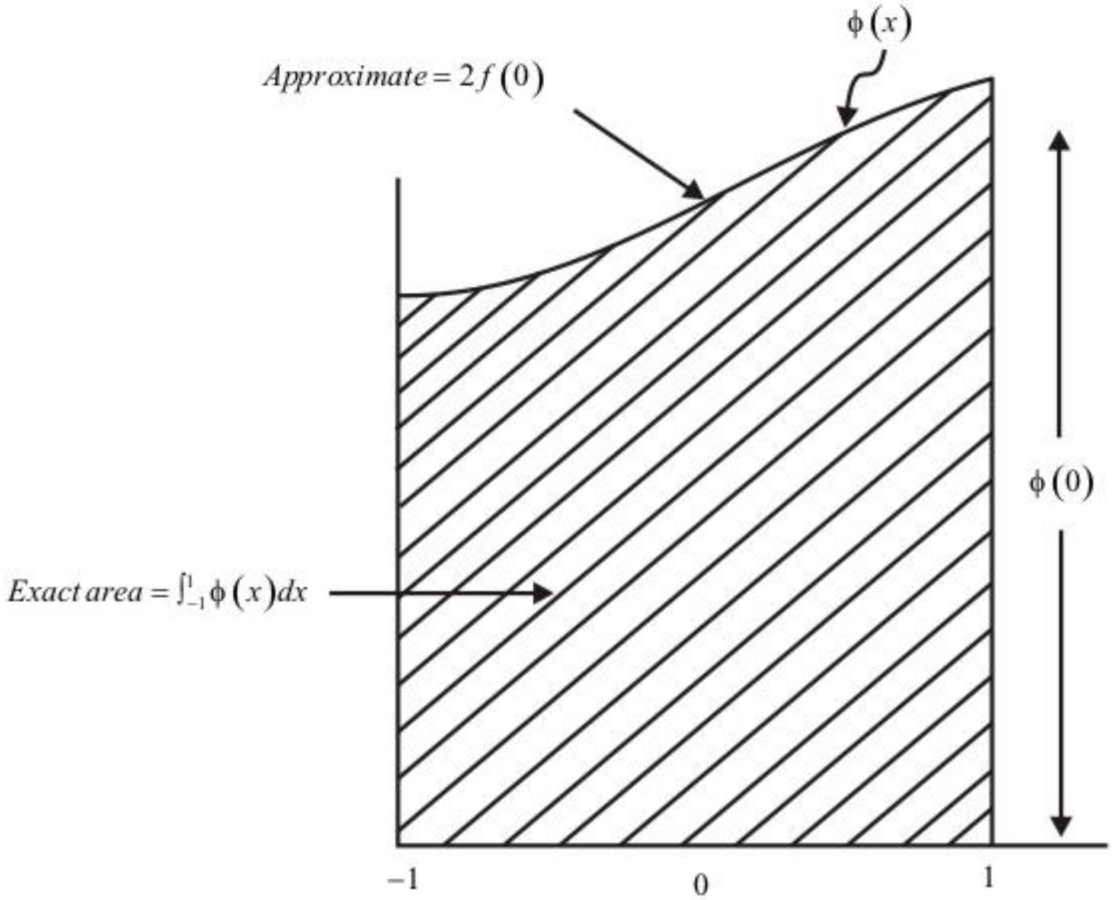
This means we have four parameters to evaluate. Hence we need a 3rd order polynomial for φ(ξ) to exactly evaluate equation

Considering, $\phi(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3$

$$\text{Error} = \left[\int_{-1}^{1} \left(a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3\right) d\xi\right] - \left[w_1\phi(\xi_1) + w_2\phi(\xi_2)\right]$$

$$\Rightarrow 2a_0 + \frac{2}{3}a_2 - w_1\left(a_0 + a_1\xi_1 + a_2\xi_1^{\,2} + a_3\xi_1^{\,3}\right) - w_2\left(a_0 + a_1\xi_2 + a_2\xi_2^{\,2} + a_3\xi_2^{\,3}\right) = 0$$

$$\Rightarrow (2 - w_1 - w_2)a_0 - (w_1\xi_1 + w_2\xi_2)a_1 + \left(\frac{2}{3} - w_1\xi_1^{\,2} - w_2\xi_2^{\,2}\right)a_2 - \left(w_1\xi_1^{\,3} + w_2\xi_2^{\,3}\right)a_3 = 0$$



One-point Gauss Quadrature

Requiring zero error yields

$$w_1 + w_2 = 2$$
$$w_1\xi_1 + w_2\xi_2 = 0$$
$$w_1\xi_1^2 + w_2\xi_2^2 = \frac{2}{3}$$
$$w_1\xi_1^3 + w_2\xi_2^3 = 0$$

These nonlinear equations have the unique solution as

$$w_1 = w_2 = 1 \qquad \xi_1 = -\xi_2 = -1/\sqrt{3} = -0.5773502691$$

From this solution, we can conclude that n-point Gaussian Quadrature will provide an exact solution if φ(ξ) is a polynomial of order (2n-1) or less.

Important Questions

1. Explain about Numerical Integration.

2. Write about Static condensation.

3. Explain briefly about assembly of elements.

4. Write the solution techniques for static loads.

5. Describe briefly about Geometric invariance & Displacement model.

6. Describe briefly about convergent and compatibility requirements.